Bet on a bubble asset ? An optimal portfolio allocation strategy

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May 2023
(work in progress, do not cite)

Abstract

We discuss portfolio allocation when one asset exhibits phases of locally explosive behavior. We model the conditional distribution of such an asset through mixed causal-non-causal models which mimic well the speculative bubble behaviour. Relying on a Taylor-series-expansion of a CRRA utility function approach, the optimal portfolio(s) is(are) located on the mean-variance-skewness-kurtosis efficient surface. We analytically derive these four conditional moments and show in a Monte-Carlo simulation exercise that incorporating them into a two-assets portfolio optimization problem leads to substantial improvement in the asset allocation strategy. All performance evaluation metrics support the higher out-of-sample performance of our investment strategies over standard benchmarks such as the mean-variance and equally-weighted portfolio. An empirical application on three bubble hedging portfolios that rely on CO2, Brent, and WTI price series respectively as the speculative assets confirms these findings.

Keywords: non-causal process, α-stable, asset allocation, utility function, out-of-sample performance
JEL: C51, C22, G12

1. Introduction

The question of portfolio allocation with bubble assets is a highly relevant empirical question nowadays in the context of the emergence of private pension systems, which increases competition within fund management industry by pushing more and more individuals to choosing

*We are grateful to Jean-Michel Zakoian, Christophe Pérignon, Serge Darolles, Gaëlle le Fol. We also thank the seminar participants at Dauphine University, University of Paris Nanterre, as well as the participants at the 4th Quantitative Finance and financial Econometrics (Aix-Marseille University), 16th International Conference on Computational and Financial Econometrics (King's College London) and the 21st Conference Développements Récents de l'Econométrie Appliquée à la Finance" (University of Paris Nanterre) for helpful comments and discussions.

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among funds with different characteristics, in terms of realized portfolio performance, for example.

A close look at the dynamics of various asset prices, that are sometimes called speculative assets, reveals the presence of phases of locally explosive behaviors, i.e. increasing patterns followed by a burst. Called rational asset pricing bubbles when due to rational deviations from the fundamental value [see Blanchard and Watson, 1982, Tirole, 1985], these phenomena have been detected more and more accurately in the financial markets across the world together with the more traditional properties of heavy-tailed marginal distributions and volatility clustering.

A rich theoretical literature has been focusing on two aspects of this phenomenon: the investment problem and the financial economic implications, [see e.g. Davis and Lleo, 2013]. To explain how a bubble originates in the market, researchers generally relied on standard martingale theory of bubbles [Biagini et al., 2014, Jarrow et al., 2010, Protter, 2012], or added further assumptions such as portfolio constraints or defaultable claims, [see Biagini and Nedelcu, 2015, Hugonnier, 2012, Jarrow et al., 2012]. The impact of bubbles on economic growth [Martin and Ventura, 2012, Carvalho et al., 2012] or on unemployment [Hashimoto and Im, 2016, 2019, Miao et al., 2016] has also been scrutinized recently. But this phenomenon has not been thoroughly gauged so far through the lens of portfolio allocation, although optimal portfolio selection has been a major topic in finance since the works of Markowitz [1952].

The scarcity of this literature may be explained by the distributional specificities of bubble asset prices and the risk they incur although, from a financial perspective, investors are certainly interested in constructing portfolios hedging bubble burst risk. Indeed, traditional portfolio theory is consistent with expected utility and its von Neumann-Morgenstern axioms of choice when either asset returns are normally distributed (i.e., higher moments are irrelevant), or investors have a quadratic utility function [see e.g. Samuelson, 1967]. But these assumptions were shown not to be empirically justified [see Mandelbrot, 1963, Ang et al., 2006, Massacci, 2017, Ingersoll, 1975, Scott and Horvath, 1980, among others].

This lead researchers and practitioners to intensively work on new portfolio allocation strategies, which, among others, pay attention to higher order moments, namely asymmetry and fat-tailness [see Briec et al., 2013, Kolm et al., 2014] for literature reviews). A wide variety of approaches have been proposed in this literature: Taylor expansion of the expected utility [Jondeau and Rockinger, 2006, 2012, Guidolin and Timmermann, 2008, Martellini and Ziemann, 2010], Gram-Charlier expansion of downside risk measures [Favre and Galeano, 2002, León and Moreno, 2017, Zoia et al., 2018, Lassance and Vrins, 2021], the shortage function of [Briec et al., 2007, 2013, to name but a few].

A portfolio strategy based on an accurate characterization of the mechanism generating financial bubbles seems necessary to avoid misleading outcomes. However, neither ARMA nor (G)ARCH / stochastic volatility models, traditionally used to characterize the predictive distribution of returns, are able to mimic such bubble behaviours. To our knowledge, only the paper by Ghahtarani [2021] discusses portfolio allocation in bubble conditions. The author introduces a new portfolio risk measure and shows that it can perform better than classical risk measures in bubble situations. To this aim, he uses a fuzzy neural network model to compute scenario paths of end-horizon market value. But the approach is not specifically designed for portfolio allocation, it operates in multiple steps and requires accounting for uncertainty surrounding the fundamental and market value predictions.

We contribute to this literature by documenting the attractiveness of portfolio strategies that account explicitly for the distributional characteristics of bubble assets. More precisely, we exploit very recent theoretical results on non-causal models to appropriately characterize the conditional distribution of asset prices exhibiting bubble behaviour. Indeed, non-causal autoregressive processes with stable distributed errors appear to be fit to model speculative financial bubbles as they mimic well locally explosive patterns [see e.g. Gourieroux and Zakoian, 2017].

Our approach is anchored in the classical theoretical rational-expectations bubble framework proposed by Blanchard and Watson [1982]. A bubble occurs when prices temporarily deviate from the fundamental value. But if Blanchard and Watson's model features successive bubble/burst cycles, the non-causal model may generate more realistic price dynamics where bubble events intersperse calmer periods. Besides, the gradual collapse in the dynamics of mixed causal-noncausal model (hereafter MAR) reconciles the rational expectations bubbles with regular variation tail indexes above 1, a well-documented statistical property of financial data, [see Lux and Sornette, 2002].[4] Most importantly, they exhibit surprising features such as a predictive distribution with lighter tails than the marginal distribution, which allows one to obtain predictions of

[4]We defer the reader to Fries [2021] for further discussion on the link between non-causal models and rational bubbles à la Blanchard and Watson [1982].

higher-moments that are expected to be of crucial importance for the (non-)investment decision. Indeed, this framework relaxes the finite variance constraint while insuring the stationarity of the process, [see Gourieroux et al., 2020, on the existence of multiple stationary nonlinear equilibria in bubble models].

By relying on the results of Fries [2021], we derive the first four conditional moments of an α-stable MAR(1,1) process and show that incorporating them into a two-assets portfolio optimization problem can lead to substantial improvement in the asset allocation strategy. For this, we consider the standard Taylor-series-expansion of a CRRA utility function approach à la Jondeau and Rockinger [2006, 2012], [see also Martellini and Ziemann, 2010]. The optimal portfolio(s) is(are) located on the mean-variance-skewness-kurtosis efficient surface in the sense that no other portfolio can dominate it on all four moments. But since there is evidence that standard utility functions are locally quadratic and higher-order moments may not significantly impact portfolio selection [see e.g. Markowitz, 2014], we also consider, as a robustness check, a polynomial-goal-programming (PGP) problem so as to find a portfolio on the higher-moment efficient surface without the need to specify a utility function. An advantage of our bubble portfolio optimisation approach (hereafter BP) over the benchmarks is that the optimal strategies take the form of couples – investment share and investment horizon –. The endogenous character of the latter leads to fewer rebalancings over the global investment horizon, simplify portfolio management operations relative to the daily rebalancing approach of the benchmarks. The economic value of our strategy is compared with standard benchmarks such as the mean-variance and equally-weighted portfolios.

In contrast to Ghahtarani [2021], if his machine learning framework were to be used for portfolio allocation, our approach is not scenario-depedent. Besides, the non-causal framework also presents the advantage of ease of interpretability, in the sense that the solution(s) of the portfolio allocation problem can be traced back to the conditioning value of the bubble asset dynamics and the higher-order moments of the conditional return distribution.

A set of Monte-Carlo simulations emphasizes the reliability of our BP approach. Indeed, the portfolio strategies based on estimates of the MAR(1,1) parameters are clustered around the theoretical optimal ones, i.e. based on the true parameters, and their dispersion reduces quickly as the sample size increases. This indicates that estimation uncertainty does not affect much the portfolio allocation problem. Note, however, that the starting value of the speculative asset $X_t = x$

matters a lot in the selection of the optimal investment share and horizon, which is not the case of the no-bubble asset. This is expected to lead to investment strategies that outperform standard benchmarks such as the mean-variance and equally-weighted portfolios. We study this intuition by simulation and rely on several performance evaluation measures to gauge the out-of-sample relative performance of our approach. All methods used, i.e. terminal wealth, opportunity cost, Sharpe and modified Sharpe ratios, support the superiority of our portfolio allocation strategy. The difference is particularly significant when the conditioning values $X_t = x$ are in the tails of the marginal distribution of the process, i.e. when the first asset is indeed close to the peak of a bubble period, which is of outmost importance for the investor.

An empirical application on three bubble hedging portfolios that use the CO2, the Brent and the WTI prices respectively, as speculative assets confirms these simulation-based results. As a preliminary step to select candidate assets, we test for the presence of bubbles in asset price dynamics by relying on the recent generalized-sup ADF test of Phillips et al. [2015] that is appropriate for rational bubble frameworks, among others.[5] The pseudo-out-of-sample performance of our allocation approach is then compared to that of the two benchmark models for each of the three hedging porfolios considered. A battery of robustness checks is performed and all findings support the superiority of our BP approach to the benchmarks whatever the investor's allocation program and preferences specification. A more realistic setting that accounts for transaction costs is also discussed as well as the case where portfolio optimization relies on the first two moments of the price distribution. All in all, the more important the non-causal dynamics, i.e. the more anticipative the bubble behaviour of the speculative series, the stronger the possibilities of favourably hedging it using the BP approach.

The paper is structured as follows. In Section 2 we introduce the proposed allocation problem. Section 3 summarizes a Monte Carlo study that discusses the impact of parameter uncertainty on the optimal strategies. In Section 4 we conduct a simulation-based out-of-sample horce-race with standard benchmarks to evaluate the relative economic value of our approach, while Section 5 details the empirical application. Finally, Section 6 concludes and the Appendices include proofs of results and additional empirical findings based on the PGP allocation framework.

[5]Early tests for rational bubbles relied on Shiller [1981]'s variance bounds test, West [1987, 1988]'s two step procedure or cointegration tests, but these approaches are subjected to multiple issues.

2. Bubble-riding allocation problem

Hedging bubble asset risk is nowadays a particularly important issue for an investor handling speculative assets. In this section, we provide a unified framework to solve the allocation problem in presence of an asset exhibiting a bubble behaviour. First, we formally introduce the portfolio allocation problem and then provide the necessary quantities to compute the conditional moments of portfolio return distribution when the speculative asset price is modeled as a mixed causal-noncausal process. Finally, we briefly review the methods that will be used to evaluate the economic value of the optimal portfolio strategies.

2.1. *Optimal portfolio allocation*

We investigate the asset allocation problem in the context with a speculative asset price X_t, for which the dynamics of higher order conditional moments is of particular importance, and a bubble-free one, S_t. Two approaches that account for higher-order moments in the choice of the optimal portfolio have gained investors' attention to date and are considered in our analysis. The first is based on a Taylor expansion of the expected utility function, while the latter consists in the Polynomial Goal Programming (PGP) model. We privilege the CRRA utility function because it is probably economically the most relevant preference family, as it realistically assumes that risk aversion is relatively constant over wealth levels, [see also Jondeau and Rockinger, 2006, 2012, and references therein]. A complementary analysis, based on the PGP approach, is available in Appendix B [see De Athayde and Flôres Jr, 2004, for arguments in favour of this approach].

We consider an investor endowed with wealth W_t at present date t, who allocates her portfolio constituted of these two assets to maximize the expected utility $U(W)$ over her end-of-period wealth W_{t+H}. The initial wealth is innocuous to the optimization problem and arbitrarily set to one. The investor has an investment horizon H: at date t, she will decide of the share ω (resp. $1-\omega$) to invest in the speculative asset (resp. bubble-free asset), and of the intermediate horizon $h \leq H$ at which she commits to liquidate its holding of speculative asset and to invest the proceedings in the bubble-free asset until $t+H$. Short selling is allowed, hence portfolio weights can take both positive and negative values. This leads to an optimization problem of the terminal wealth W_{t+H} or, equivalently, of the overall return $R_{t+H} = (W_{t+H} - W_t)/W_t$ in both the allocation ω and the intermediate horizon h, which is new in the financial literature.

We assume that the speculative asset's price X_t follows a mixed causal-noncausal stable AR process, i.e. MAR(1,1), with a non-zero location parameter. This choice is motivated by the recent econometric literature that proved non-causal models to be a convenient way to model locally explosive phenomena such as speculative bubbles, while featuring heavy-tailed marginals and conditional heteroscedastic effects generally encountered in financial data [see e.g. Cavaliere et al., 2020, Fries and Zakoian, 2019, Gourieroux and Jasiak, 2018, Gourieroux and Zakoian, 2017]. The bubble-free asset is assumed to follow a Geometric Brownian Motion (GBM) dynamics with drift υ and volatility ς. The price processes (X_t) and (S_t) will be assumed independent, which provides a nice framework for hedging purposes.[6]

For a given strategy (ω, h), the terminal wealth can be expressed as

$$W_{t+H} = \frac{S_{t+H}}{S_{t+h}} \Big(\omega \frac{X_{t+h}}{X_t} + (1-\omega) \frac{S_{t+h}}{S_t} \Big), \tag{1}$$

or alternatively, in terms of returns,

$$W_{t+H} = 1 + R_{t+H},$$

where the terminal portfolio return R_{t+H} writes

$$R_{t+H} = \Big(1 + r^S_{t+h,t+H}\Big)\Big(\omega r^X_{t,t+h} + (1-\omega) r^S_{t,t+h} + 1\Big) - 1,$$

with $r^S_{t+h,t+H} := S_{t+H}/S_{t+h} - 1$, $r^S_{t,t+h} := S_{t+h}/S_t - 1$, and $r^X_{t,t+h} := X_{t+h}/X_t - 1$, the asset's returns in-between the key investment events.

In this framework, we follow Jondeau and Rockinger [2006, 2012] to approximate the allocation problem.[7] The CRRA utility maximization program of the fourth order Taylor approximation

[6] In this framework, the independence hypothesis does not appear as a strong assumption. As we focus on investment during periods where an asset price exhibits a bubble behaviour, its dynamics cannot be correlated over this time-interval with that of a safe(r) asset. In practice it is reasonable to think of the second *asset* as a well-diversified portfolio whose constituents do not exhibit any bubble behaviour. This makes the second asset an attractive hedge against the risk of bubble collapse in the first asset.

[7] Lhabitant et al. [1998] has shown that the infinite Taylor series expansion converges to the expected utility in the CRRA case for wealth levels between 0 and $2\overline{W}$ that appear to be large enough for stocks and bonds regardless of the degree of non-normality, in particular when short-selling is prohibited.

around the expected terminal wealth is

$$\max_{(\omega,h)} \mathbb{E}[U(W_{t+H}|X_t,S_t)] \approx \sum_{k=0}^{4} \frac{U^{(k)}(\overline{W}_{t+H})}{k!} \mathbb{E}\Big[(W_{t+H} - \overline{W}_{t+H})^k | X_t, S_t\Big], \tag{2}$$

with $U(c) = c^{1-\gamma}/(1-\gamma)$ for a risk aversion parameter $\gamma > 0$ and $\overline{W}_{t+H} = \mathbb{E}\Big[W_{t+H}|X_t,S_t\Big]$. The investor's preference (or aversion) toward the k^{th} moment is directly given by the k^{th} derivative of the utility function. The effects of the third and fourth moments on the approximated expected utility are positive and negative, respectively, and correspond to financial theory [see Scott and Horvath, 1980]. The expected utility also depends on the central conditional moments of the distribution of terminal wealth, which can be expressed in terms of conditional moments of the portfolio return distribution as $\mathbb{E}\Big[(W_{t+H} - \overline{W}_{t+H})^k|X_t,S_t\Big] = \mathbb{E}\Big[(R_{t+H} - \overline{R}_{t+H})^k|X_t,S_t\Big]$, since $\overline{W}_{t+H} = 1 + \overline{R}_{t+H}$ with $\overline{R}_{t+H} := \mathbb{E}[R_{t+H}|X_t,S_t]$. It is just a matter of algebra using the independence between (X_t) and (S_t) to express the objective functions in terms of the conditional moments of the speculative asset price, $\mathbb{E}\Big[X^p_{t+h}|X_t\Big]$, $p = 1,2,3,4$, that are detailed in the next subsection, and the parameters (see Appendix A.1 for further computational details).

2.2. *Conditional moments of MAR(1,1) α-stable processes*

In this subsection we discuss the existence and derivation of the first four conditional moments of the speculative asset price. As the econometric literature has identified MAR processes to be appropriate for financial bubble modelling, we rely on them, [see e.g. Hecq and Voisin, 2021, and references therein].[8]

Let (X_t) be the α-stable solution of the MAR(1,1) process $X_t = \varphi^\circ X_{t+1} + \varphi^\bullet X_{t-1} + \varepsilon_t$, with i.i.d. α-stable errors, $\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha,\beta,\sigma,\mu)$ and $\alpha \neq 1$ (for simplicity), $\beta \in [-1,1]$, and $\sigma > 0$. The process is well defined and strictly stationary for $|\varphi^\circ| < 1$, $|\varphi^\bullet| < 1$, and $\varphi^\circ \neq \varphi^\bullet$. It then has a $MA(\infty)$ representation $X_t = \sum_{k\in\mathbb{Z}} a_k \varepsilon_{t+k}$, whose coefficients satisfy $\sum\limits_{k\in\mathbb{Z}} |a_k|^s < +\infty$ for some $s \in (0,\alpha) \cap [0,1]$. Without loss of generality, in the following we assume that the shift μ is null, but in practice we handle the possibility of $\mu \neq 0$ by relying on a simple transformation of the conditional moments obtained with zero location parameter to those associated with a non-null

[8] More generally, any MARMA model could be used, but this would engender a cost related to the numerical approximation of the $MA(\infty)$ coefficients [see Fries, 2021]. We prefer a more parsimonious approach for which we can obtain the formulas for the coefficients in closed form.

shift [see Section 2 in Fries, 2021].

Now let $\mathbf{X}_t = (X_t, X_{t+h})$ denote the bivariate stable vector obtained from X_t for horizon $h \geq 1$. Proposition 3.1 *i)* in Fries [2021] then applies and states the condition of existence of higher-order conditional power moments, although the marginal variance of the process X_t is infinite. In particular, the conditional moments up to integer order p, $\mathbb{E}[|X_{t+h}|^p|X_t]$, may exist as long as $\nu \geq 0$ exists such that $\sum_{k\in\mathbb{Z}} (a_k^2 + a_{k-h}^2)^{\frac{\alpha+\nu}{2}} |a_k|^{-\nu} < +\infty$ and $0 \leq p < \min(\alpha+\nu, 2\alpha+1)$, where $(a_k,\, a_{k-h})$ are the coefficients of the infinite moving average representation of the process. The more *anticipative, i.e. noncausal* the process, the larger $\nu \geq 0$, which insures the existence of all conditional moments up to order $2\alpha+1$ at all prediction horizons when (a_k) decays geometrically or hyperbolically for example.

Proposition 1. *For $\alpha \neq 1$, the moments $\mathbb{E}[X_{t+h}^p|X_t]$, $p \leq 4$, when they exist, are given by Theorems 2.1 and 2.2 in Fries [2021] as a function of four quantities, σ_1^α, β_1, κ_p, and λ_p and a family of functions $\mathcal{H}$. We demonstrate that in the case of a MAR(1,1) process these constants are equal to*

$$
\begin{aligned}
\sigma_1^\alpha =& \sigma^\alpha \frac{1-|\varphi^\circ\varphi^\bullet|^\alpha}{(1-\varphi^\circ\varphi^\bullet)^\alpha(1-|\varphi^\circ|^\alpha)(1-|\varphi^\bullet|^\alpha)},\\
\beta_1 =& \beta \frac{1-\varphi^{\circ<\alpha>}\varphi^{\bullet<\alpha>}}{1-|\varphi^\circ|^\alpha|\varphi^\bullet|^\alpha}\frac{(1-|\varphi^\circ|^\alpha)(1-|\varphi^\bullet|^\alpha)}{(1-\varphi^{\circ<\alpha>})(1-\varphi^{\bullet<\alpha>})},\\
\kappa_p =& \frac{\varphi^{\bullet hp}(1-|\varphi^\circ|^\alpha)+(\varphi^{\circ-hp}|\varphi^\circ|^{h\alpha})(1-|\varphi^\bullet|^\alpha)}{1-|\varphi^\circ\varphi^\bullet|^\alpha}\\
&+\frac{(\varphi^{\bullet hp}|\varphi^\circ|^\alpha(\varphi^\bullet\varphi^\circ)^{-p}-\varphi^{\circ-hp}|\varphi^\circ|^{\alpha h})(1-|\varphi^\circ|^\alpha)(1-|\varphi^\bullet|^\alpha)}{(1-|\varphi^\circ|^\alpha(\varphi^\bullet\varphi^\circ)^{-p})(1-|\varphi^\circ\varphi^\bullet|^\alpha)},\\
\lambda_p =& \frac{\varphi^{\bullet hp}(1-|\varphi^\circ|^{<\alpha>})+(\varphi^{\circ-hp}\varphi^{\circ<\alpha>h})(1-\varphi^{\bullet<\alpha>})}{1-\varphi^{\circ<\alpha>}\varphi^{\bullet<\alpha>}}\\
&+\frac{(1-\varphi^{\circ<\alpha>})(1-\varphi^{\bullet<\alpha>})}{1-\varphi^{\circ<\alpha>}(\varphi^\bullet\varphi^\circ)^{-p}}\frac{(\varphi^{\bullet hp}\varphi^{\circ<\alpha>}(\varphi^\bullet\varphi^\circ)^{-p}-\varphi^{\circ-hp}\varphi^{\circ<\alpha>h})}{1-\varphi^{\circ<\alpha>}\varphi^{\bullet<\alpha>}},
\end{aligned}
$$

where $y^{<\alpha>} = \text{sign}(y)|y|^\alpha$ for any $y \in \mathbb{R}$. σ_1 and β_1 denote the scale and asymmetry parameters of the marginal distribution of X_t, whereas the constants κ_p and λ_p, $p > 2$, generalize standard dependence measures invoked in the literature to powers of X_t and X_{t+h} in the asymmetric case. At the same time, $\mathcal{H}$ contains functions related to the marginal density of the stable random variable X_t and for $n \in \mathbb{N}$, $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}$, $x \in \mathbb{R}$ is defined as

$$
\mathcal{H}(n,\theta;x) = \int_0^{+\infty} e^{-\sigma_1^\alpha u^\alpha} u^{n(\alpha-1)} \big(\theta_1 \cos(ux - \alpha\beta_1\sigma_1^\alpha u^\alpha) + \theta_2 \sin(ux - \alpha\beta_1\sigma_1^\alpha u^\alpha)\big) du.
$$

Proof. See Appendix A.2 □

Remark 1. *The conditional moments can be easily computed for $p \leq 4$ and $h \geq 1$ once the functions $\mathcal{H}(n, \theta; x)$ are evaluated for $n = 2, 3, 4$ by following the approach discussed by Fries [2021] and originally proposed by Samorodnitsky et al. [1996] for the conditional expectation.*

Remark 2. *The asymptotic expressions for the conditional moments with respect to the conditioning variable, i.e. when X_t becomes large, given in Proposition 2.1 of Fries [2021] and stated below, remain valid in the MAR(1,1) case when σ_1^α, β_1, κ_p, and λ_p are replaced by the expressions given in Proposition 1 above. To be more precise, if the conditional moment of order p of a bivariate α-stable vector exists and $|\beta_1| \neq 1$, then,*

$$x^{-p}\mathbb{E}[X_{t+h}^p | X_t = x] \underset{x\to\infty}{\to} \frac{\kappa_p + \lambda_p}{1 + \beta_1}, \quad x^{-p}\mathbb{E}[X_{t+h}^p | X_t = x] \underset{x\to-\infty}{\to} \frac{\kappa_p - \lambda_p}{1 - \beta_1}. \tag{3}$$

2.3. *Performance evaluation measures*

Several investment ratios, e.g. the Sharpe, Sortino, and Omega ratios, and relative performance measures, e.g. the opportunity cost or performance fee (OC) and the Graham–Harvey metric, have been used in the literature to evaluate portfolios' performance, [see e.g. Jondeau and Rockinger, 2006, 2012, González-Pedraz et al., 2015]. But since standard ratios ignore investors' positive preferences for odd moments and aversion to even moments, they are not appropriate for investments with non-normal returns. Several alternatives have been proposed, such as the modified Sharpe ratio (mSharpe) of Favre and Galeano [2002], Gregoriou and Gueyie [2003], which uses as a risk measure an estimator for Value-at-Risk based on the Cornish–Fisher expansion and the first four moments of the return distribution.

In this paper we employ the OC measure to evaluate the out-of-sample performance of our strategy relatively to two traditional benchmarks, the equally-weighted portfolio (*EW*) and the standard mean-variance (MV) portfolio. This corresponds to the amount that needs to be added to the return of a competing benchmark strategy so that the investor becomes indifferent to the portfolio decision based on our framework. We also report the mSharpe ratio and, for comparison reasons, the Sharpe ratio, which indicate the risk premium relatively to not investing at all (the risk free rate is assumed to be null). Finally, we check whether our approach performs better by testing the equality of the medians of the terminal wealth and investment ratios of the strategies over the out-of-sample.

3. Monte Carlo experiments

As the investor does not have perfect knowledge of the parameters of the distribution of the speculative asset, we investigate the impact of parameter estimation on portfolio allocation in a Monte-Carlo experiment. We adopt a parametric plug-in estimation approach and proceed in two steps.[9] First, we gauge the sensitivity of the conditional moments of returns to parameter estimation and then we look into the variability this induces in the optimal portfolio strategy.

We simulate $M = 2000$ trajectories of $N = \{250, 1000, 5000\}$ observations from the MAR(1,1) process $(1 - 0.9F)(1 - 0.1B)X_t = \varepsilon_t$ where $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.3, 1.5, 15)$. We then estimate the conditional power moments by replacing the theoretical constants $\sigma_1^\alpha,\ \beta_1,\ \kappa_p,\ \lambda_p.$ in Proposition 1 by their empirical counterparts computed by plugging-in the MAR(1,1) parameter estimates obtained by Maximum Likelihood.[10]

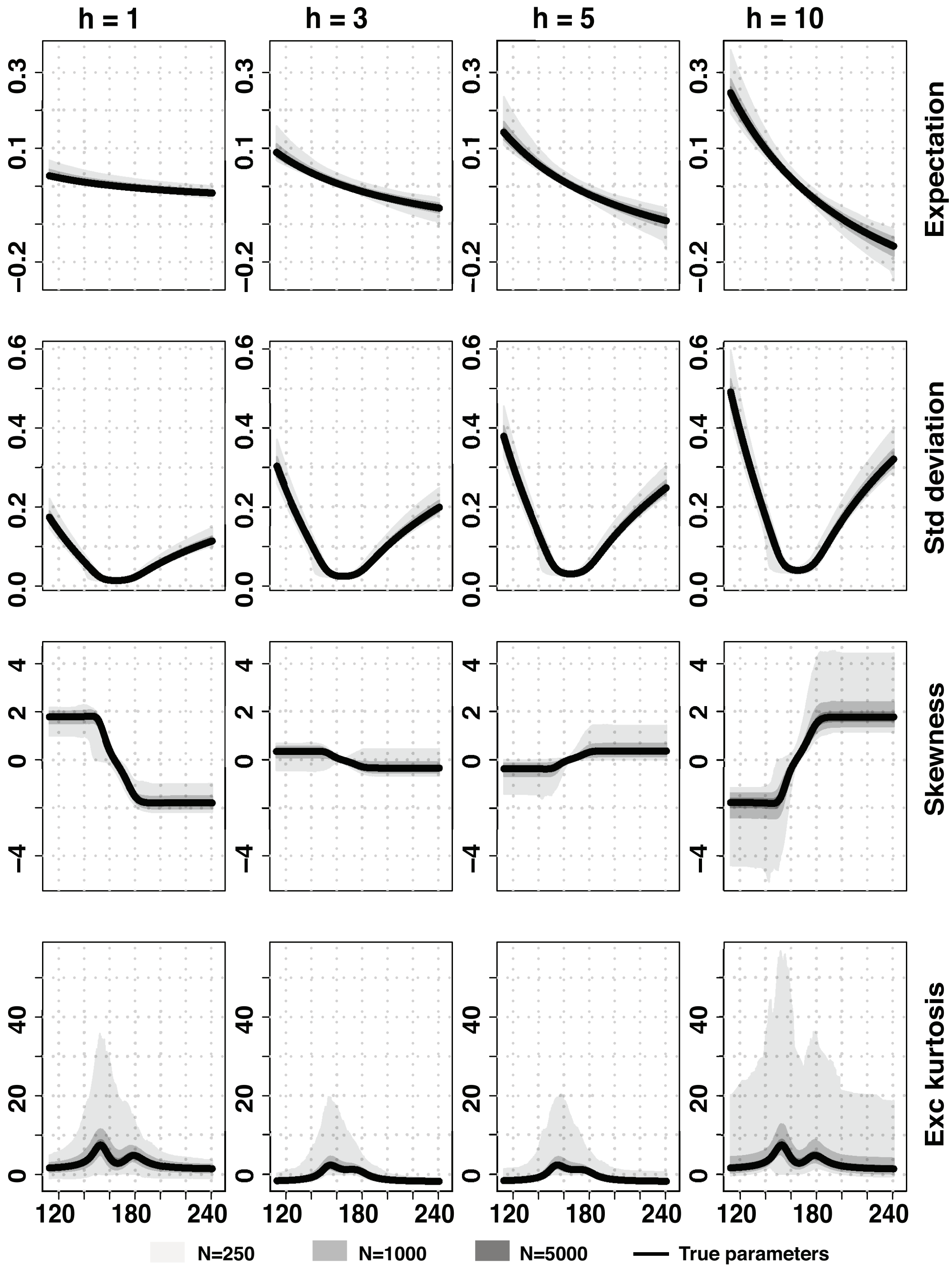
The results are displayed in Figure 1 for prediction horizons $h = 1, 3, 5, 10$ and conditioning values $x \in (112 - 245)$ that correspond to the 0.05% and 99.95% quantiles of the marginal distribution of X_t. These results take the form of a pointwise 5% - 95% interquartile interval of the conditional moment estimators for each sample size N. Notice that the theoretical conditional moments, based on the true values of the parameters and represented by a black line, always belong to the empirical interquartile range. More precisely, the interquantile intervals are narrow around most of the true conditional moments curves, even for small sample sizes. They are larger for higher-order moments and large horizons when $N = 250$ but narrow down fast as the sample size increases. Overall, the plug-in method appears to be a good way to estimate the conditional moments even when the conditioning values $X_t = x$ are in the tails of the marginal distribution of the process.

In the second step we hence investigate the impact of parameter estimation on the selected

[9] A model-free non-parametric approach could also be envisaged, but it would engender a dramatic loss in efficiency, especially for conditioning values $X_t = x$ far away from the central values of the process (X_t), [see Fries, 2021, Supplementary Material].

[10] To facilitate the estimation, we initialize the parameters of the α stable distribution by relying on the approach of McCulloch [1986]. Provided the ML estimator is consistent, which is the case for the one used here [see Andrews et al., 2009], the plug-in estimators of the conditional moments will also be consistent.

Figure 1: Conditional moments of return distribution



Notes: First four conditional moments of returns when the price series follows a MAR(1,1) process $(1-0.9F)(1-0.13B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.3, 1.5, 15)$. The conditional moments are obtained for conditioning values $X_t = x \in (112, 245)$, i.e. 99.9% of the probability mass of the marginal distribution of X_t is supported on this interval. The black line represents the theoretical moments, whereas the gray shaded areas correspond to the simulated conditional moments based on 2000 draws from the above α-stable distribution for the three sample sizes. In the latter case the MAR(1,1) parameters are estimated by Maximum Likelihood and plugged in the formulae of Proposition 1. The results are displayed for three horizons, $h = 1, 3, 5, 10$ and three sample sizes, $N = 250, 1000, 5000$.

portfolios. The simulated conditional moments of returns obtained from the ML estimates of the MAR(1,1) process are plugged in the CRRA portfolio optimization program to get the optimal portfolio strategi(es) in the form of couples (ω^*, h^*), which define the part of the wealth to invest in the bubble asset and the horizon of this investment given that the overall investment horizon is fixed to $H = 26$ periods, i.e., a year of daily trading activity. To be more precise, we search for optima (ω^*, h^*) in the set $[-1, 1] \times [0, 250]$, thus allowing short strategies. As the optimization program is likely non-convex, several strategies may lead to the same terminal wealth, and in this case they are all labeled as optimal strategies. We round ω^* to the closest percentage point and report h^* in weeks. Besides, by convention, if either $\omega^* = 0$ or $h^* = 0$, we report $(\omega^*, h^*) = (0, 0)$. For the bubble-free asset, we set v and ς so that the annual return and volatility equal 2%.

Figures 2 and 3 propose a visualization of the optimal investment strategies if the DGP were known and of the impact of parameter estimation on the selected optimal portfolios for a CRRA investor with risk aversion parameter $\gamma = 10$. For each starting value of the speculative asset $X_t = x$ defined by a specific quantile of its distribution and each sample size N, we plot the mass repartition of the estimated strategies across the 2000 simulations in the share-horizon space. The bigger and redder the dots, the larger the mass of portfolios falling in that area. Roughly speaking, a red circle corresponds to more than 1000 identical strategies, a violet one indicate more than 500 identical ones, whereas the smallest blue dots represent between 5 and 50 identical strategies.[11] The optimal strategies under the hypothesis that the investor knows the parameters of the speculative asset dynamics are denoted by black target symbols.

While the starting value of S_t does not matter, the starting value of X_t deeply modifies the investment landscape. The first figure looks into the case of conditioning values in the lower conditional quantiles of X_t. The CRRA investor bets on a rising value of the speculative asset and opts for a full investment in it ($\omega^* = 1$) over horizons of up to fifteen weeks ahead, $h^* < 15$. This long strategy is the only optimal portfolio allocation in this setup, i.e., the equilibrium is unique for X_t outside the trough (0.0001 quantile). The optima from the simulated strategies, denoted by colored dots, are generally concentrated in the vicinity of the true optimal strategies, which indicates that estimation uncertainty does not affect much the portfolio allocation problem. As

[11] We do not report the precise values as they vary from one subplot to another due to the multiple equilibrium issue discussed earlier and the plot would become too dense to be easily readable.

the sample size N increases, the estimation becomes even more accurate and more mass gathers around the true optima.

The second figure depicts the case of conditioning values at the median and in the upper conditional quantiles of X_t. The long strategy, characterized by a share close to 1 invested over very short intervals, is optimal as we move from the center of the distribution towards the bubble zone.

Multiple optimal strategies arise as we get to the steepest part of the inflation phase of the bubble. This comes in hand with different investors betting on different scenarios according to their risk adversity. Shorting the bubble over a 2-period horizon appears to be as optimal as investing a certain share of wealth over a horizon between 2 and 7 periods. Next, in the explosive regime of the 0.99 quantile the optimal strategy is to completely short the bubble asset over a one-period horizon. Finally, above the 0.99 quantile, i.e. as the explosive regime becomes more evident, the optimal strategy consists in a fair short position over 12 to 14 periods, which is consistent with the increasing bubble crash risk.

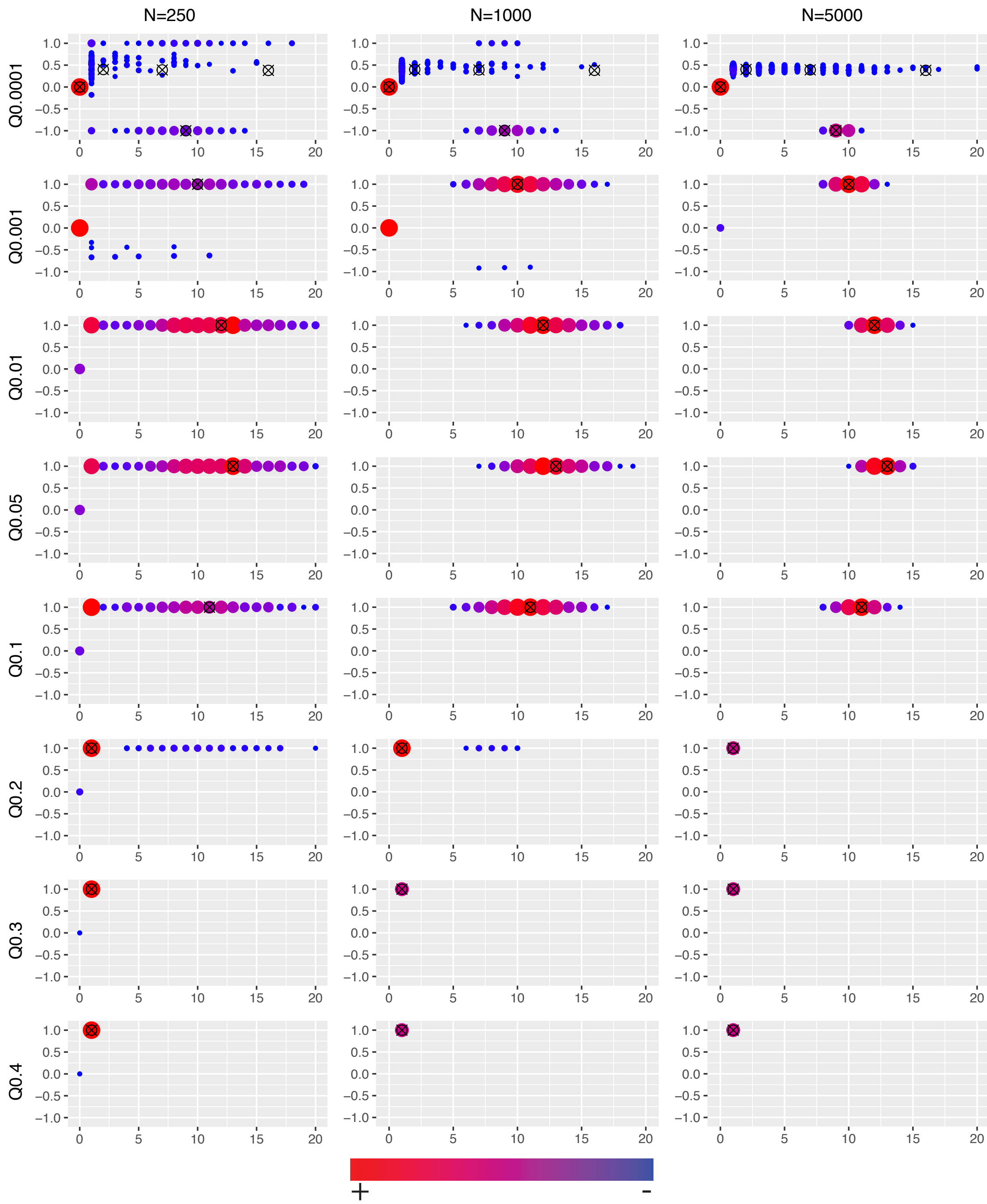
Additionally, the dispersion of simulation-based strategies around the true ones rapidly shrinks with the sample size, suggesting that the true optima can indeed be consistently retrieved after parameter estimation. For the quantiles furthest in the tail, the dispersion is in the horizon dimension rather than in the share dimension. Estimation uncertainty on the verge of a bubble crash thus mainly impacts the holding horizon. The results are robust to the choice of the risk-aversion parameter and to changes in the speculative asset price data generating process.[12]

4. Economic Value

In this section, we illustrate the usefulness of the BP approach to provide high-performing portfolio allocation strategies. As discussed previously, in our framework, the optimal investment strategies vary according to the conditioning values $X_t = x$ of the marginal distribution of the process. For this reason, they are expected to outperform standard mean-variance and equally-weighted portfolios, that cannot take into account the current state of the nature at the moment of

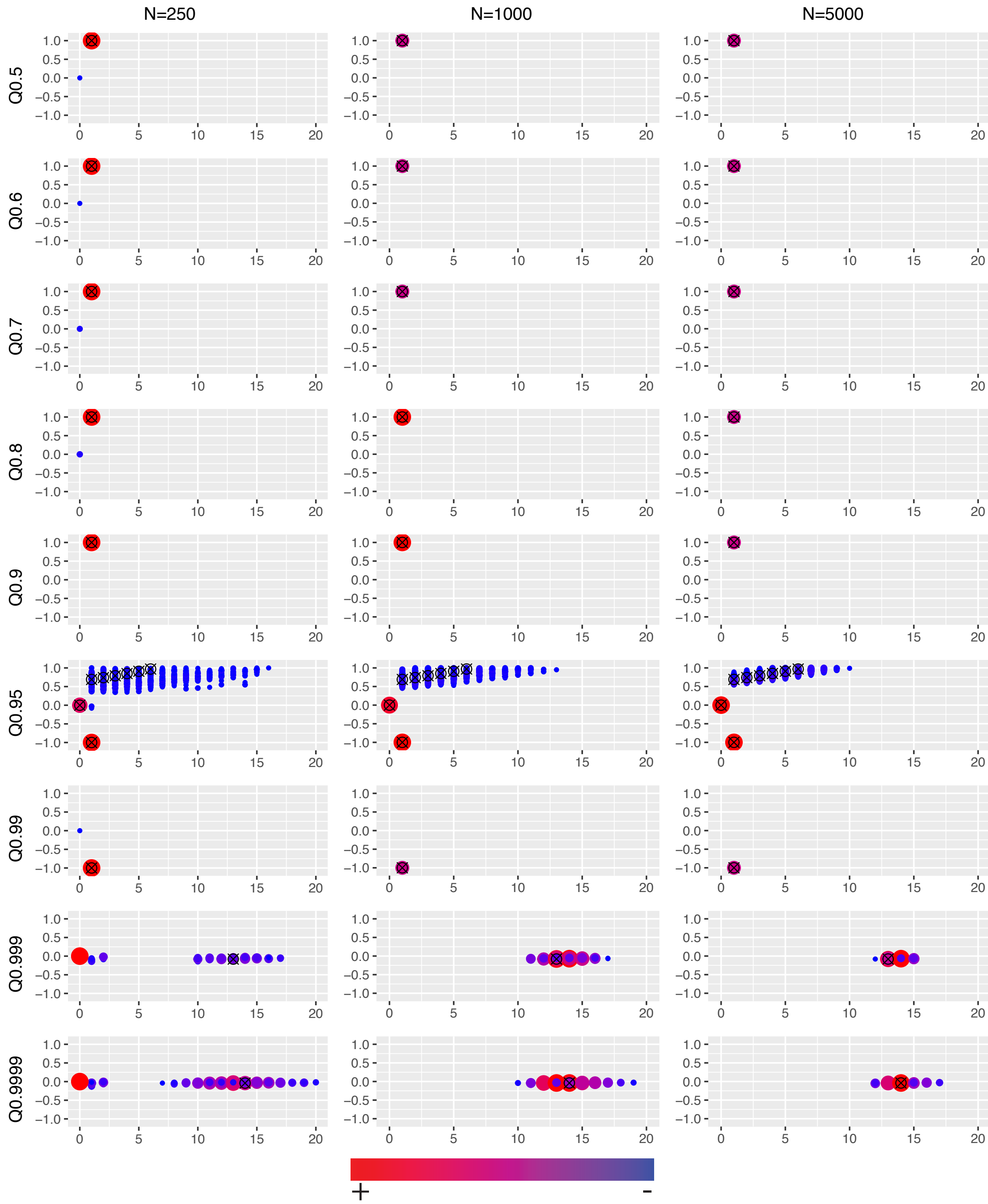
[12]The results are qualitatively similar to those obtained when using a non-causal $AR(1)$ as a DGP, but the latter seems to be quite restrictive in practice as it imposes a sudden crash of the bubble. We prefer the more general $MAR(1,1)$ specification and accept a loss in efficiency in the case where the causal parameter should actually be null.

Figure 2: Optimal portfolio strategies (lower quantiles)



Notes: Mass repartition of the optimal portfolio strategies for the CRRA utility function with $\gamma = 10$ when the speculative asset's parameters are estimated by ML across 2000 simulated trajectories of length $N = 250, 1000, 5000$ trading days and for several starting values defined by the quantiles, $Q_{\cdot}$, of the true marginal distribution of X_t. The DGP for the speculative asset price is a MAR(1,1) process $(1 - 0.9F)(1 - 0.13B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.3, 1.5, 15)$. The results are displayed in the share (vertical axis) - horizon (horizontal axis) space. The larger and redder the dots, the bigger the proportion of selected portfolios falling in that area across the 2000 simulations. A black target symbol indicates a *true* optimal portfolio, i.e. obtained for the true values of the parameters.

Figure 3: Optimal portfolio strategies (upper quantiles)



Notes: Mass repartition of the optimal portfolio strategies for the CRRA utility function with $\gamma = 10$ when the speculative asset's parameters are estimated by ML across 2000 simulated trajectories of length $N = 250, 1000, 5000$ trading days and for several starting values defined by the quantiles, $Q_{\cdot}$, of the true marginal distribution of X_t. The DGP for the speculative asset price is a MAR(1,1) process $(1 - 0.9F)(1 - 0.13B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.3, 1.5, 15)$. The results are displayed in the share (vertical axis) - horizon (horizontal axis) space. The larger and redder the dots, the bigger the proportion of selected portfolios falling in that area across the 2000 simulations. A black target symbol indicates a *true* optimal portfolio, i.e. obtained for the true values of the parameters.

investing. We study this intuition in the same Monte Carlo setup as in the previous section. More precisely, we generate 1000 trajectories of $N = 2000$ observations from the MAR(1,1) process.

For each trajectory, we use the first two thirds of the data, $\{1, 2, \ldots, T\}$, labeled as in-sample, to estimate the conditional moments of returns and identify the optimal investment strategies in the form of couples (ω, h) for conditioning values covering the whole marginal distribution of X_t implied by the DGP. The remaining one third of the data, $\{T + 1, \ldots, T + k, \ldots N\}$, labeled as out-of-sample, is used as conditioning values for a new investment. Said otherwise, we assume the investor wishes to invest his wealth in the two assets at a certain date, say $T + k$, within the out-of-sample period. To select the optimal share of the bubble asset in the portfolio, ω, and the duration of this risky investment, h, out of the $H = 250$ periods of the overall investment, she searches for the closest quantile of the theoretical distribution of X_t just below the actual conditioning price at the selected date. The couple(s) (ω, h) estimated in-sample for this quantile by using the $CRRA$ utility function with $\gamma = 10$ will then be used to construct the portfolio strateg(y/ies). For each strategy, the portfolio is rebalanced once, at period $T + k + h$. Consisting only in an investment in the no-bubble asset, it is then held constant up until date $T + k + H$.

For comparison reasons, we compute also the mean-variance and the equally-weighted portfolios over the same periods. In the case of the MV benchmark portfolio, we use the in-sample data to estimate the optimal investment share in the bubble asset. Then, we use it to construct a buy and hold strategy over H periods for each out-of-sample starting date $T + k$. Finally, the computation of the EW portfolio for the same investment horizon is immediate.

To compare the economic value of these strategies we rely on the methods introduced in Subsection 2.3. As our approach may lead to multiple optimal strategies for a given conditioning value, we report three statistics: the average one, labeled BP_{mean}, the one leading to the highest terminal wealth, labeled BP_{sup}, and the one leading to the lowest terminal wealth, labeled BP_{inf}.

Table 1 reports the results for the five portfolio strategies in terms of average, μ, and standard deviation, σ, of each performance measure over the 1000 simulated out-of-sample trajectories. Asteriks $(*/*)$ associated with the estimated μ of each of our strategies indicate that the mean of the performance measure is statistically different from that of the (EW/MV) portfolios according to Wilcoxon's test.

The average wealth for the three MAR(1,1)-based portfolios is similar and always well above that of the benchmark portfolios. Wilcoxon's test always rejects the null of equal averages, suggesting that our approach performs best in terms of terminal wealth. The results are similar when relying on the Sharpe ratio instead of wealth. Notice that the standard deviation is inflated in this case, but it largely diminishes when using the more appropriate modified Sharpe ratio that accounts for higher order moments of portfolio return distribution. Regardless of the measure used, our approach performs significantly better than the EW and MV ones, and this holds even in the worst case scenario, i.e. BP_{inf}. The positive averages of the opportunity cost also support these findings. A smaller amount needs to be added to the MV strategy than to the EW one to provide the same expected utility as our BP strategies.

Table 1: Relative performance of portfolio strategies

				Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW		
Wealth	μ	$1.019^{*/*}$	1.009	$1.069^{*/*}$	1.003	1.002		
	σ	0.038	0.061	0.058	0.052	0.006		
			EW vs			MV vs		
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}	
OC	μ	0.023	-0.006	0.036	0.001	0.005		0.006
	σ	0.057	0.072	0.056	0.036	0.046		0.047

Notes: Our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth, Sharpe ratio and modified Sharpe ratio. The opportunity cost (OC) relatively to the two benchmark portfolios is also provided. The results take the form of out-of-sample average and standard deviation over the 1000 simulations. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilcoxon's test of equality of medians at the 95% level relatively to each of the two benchmark strategies, EW and MV, respectively.

As our approach is specifically designed for investors that wish to take advantage of bubble periods, in Table 2 we focus on this setup. We assume that one invests only when the unconditional price process seems to exhibit a locally explosive behaviour, i.e. the conditioning values $X_{T+k} = x$ are above the 95% quantile of the theoretical distribution of the process. The average terminal wealth for our strategies are bigger than in the case when all the marginal distribution is considered, whereas that of the benchmark strategies is lower. The modified Sharpe ratio behaves similarly. The opportunity cost remains positive, relatively constant for the EW strategy and lower than in Table 1 for the MV portfolio.

All in all, these results indicate that our method may prove useful for the investor that includes

Table 2: Relative performance of portfolio strategies in positive bubble period

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV	
Wealth	μ	$1.045^{*/*}$	$1.042^{*/*}$	$1.048^{*/*}$	1.031	1.019	
	σ	0.047	0.046	0.048	0.030	0.020	
		vs EW			vs MV		
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.028	0.028	0.028	0.015	0.015	0.015
	σ	0.052	0.052	0.052	0.013	0.013	0.013

Notes: see note to Table 1. The results are based only on the cases where the investment is performed while the first asset exhibits a bubble period, i.e. $X_{T+k} = x$ is beyond the 95% quantile of the theoretical distribution of the price process.

a bubble asset in her portfolio. They hold when investigating the case of negative bubbles, i.e. looking only at conditioning values beyond the 95% quantile of the theoretical distribution of the process. Finally, they are qualitatively similar when fixing the risk-aversion parameter γ to 10.

5. Empirical application

5.1. Data and in-sample analysis

This section discusses the performance of the proposed portfolio allocation strategy on real data. We consider three candidates for the bubble asset: the price of CO2 as well as Brent and WTI oil prices. The oil prices are long established in the non-causal econometric literature as having a mixed-causal dynamics, while evidence of explosive behaviour in the CO2 price has been provided by Friedrich et al. [2019]. At the same time, we use *EUREX-Euro bund* settlement price series as the bubble-free asset on the European market and the *30-year US Treasury-bond* settlement price for the US market, i.e. in the case of the WTI series. Weekly data, ranging from 2017-01 to 2023-03, have been obtained from Refinitiv Eikon for all series and splitted into an in-sample part (2017-01 to 2021-05) and an out-of-sample one (2021-06 to 2023-03).

The results of the generalized-sup ADF test of Phillips et al. [2015] in Table 3 clearly indicate the presence of a locally explosive behaviour in the oil prices, while the CO2 is on the boundary of the significance level as the series exhibits a more pronounced bubble behaviour in the out-of-sample part.

Table 3: GS-ADF Test

	Test Stat.	Finite Sample Critical Value		
		90%	95%	99%
EU ETS CO2 price	1.481	1.489	1.711	2.351
Brent	2.071			
WTI	2.505			

Notes: Generalized-sup ADF test for the presence of multiple bubbles developed by Phillips et al. [2015]. The critical values are based on 1000 simulations.

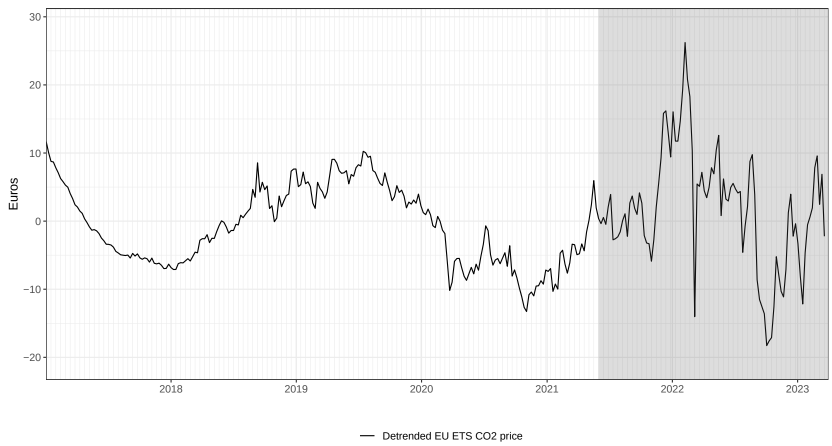
Existing econometric literature has also emphasized a certain level of deterministic nonstationarity in these series that needs to be tackled. In particular, a trending time varying fundamental part must be extracted before estimating Mixed AR models on the in-sample data.[13] Two detrending methods have gained interest in this literature. A polynomial function has been used by Hencic and Gouriéroux [2015] and by Hecq and Voisin [2019], while Hecq and Voisin [2021] use the Hodrick - Prescott filter. As the first approach is more direct to implement in the out-of-sample, we follow Hecq and Voisin [2019] and use a polynomial trend of order four to capture trending patterns in the Brent series. Similarly, a polynomial trend of order four seems to fit the CO2 series, while a polynomial trend of order three better captures the trend of the WTI series. The bubblish behaviour of the detrended series is clearly distinguishable in Figure 4, the gray region corresponding to the out-of-sample and the dense vertical grid to two-week intervals.

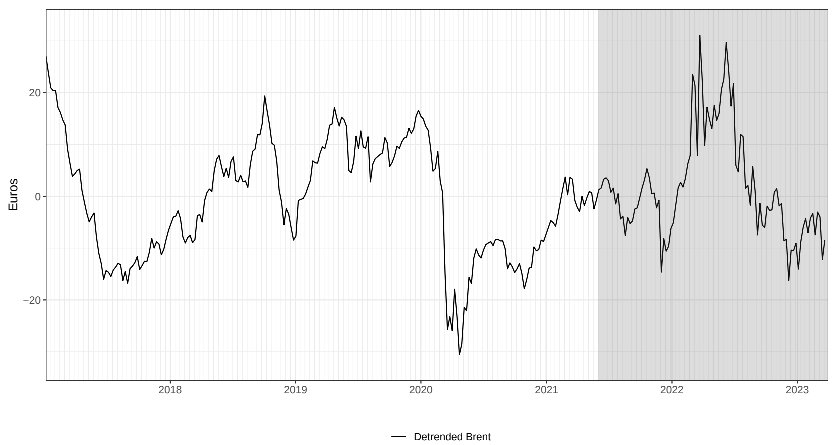
We then rely on the procedure of Lanne and Saikkonen [2011] based on the AIC information criterion to perform model selection on causal-non-causal models and identify the MAR(1,1) as the best specification for the oil series. Given the less bublish behaviour of CO2 and that a purely non-causal AR(1) model seems to fit it better, we decide to consider it as a case of a slightly misspecified data generating process in the analysis and estimate a MAR(1,1) specification as for the other two series. A 6-months investment horizon is defined in all cases by fixing $H = 26$.

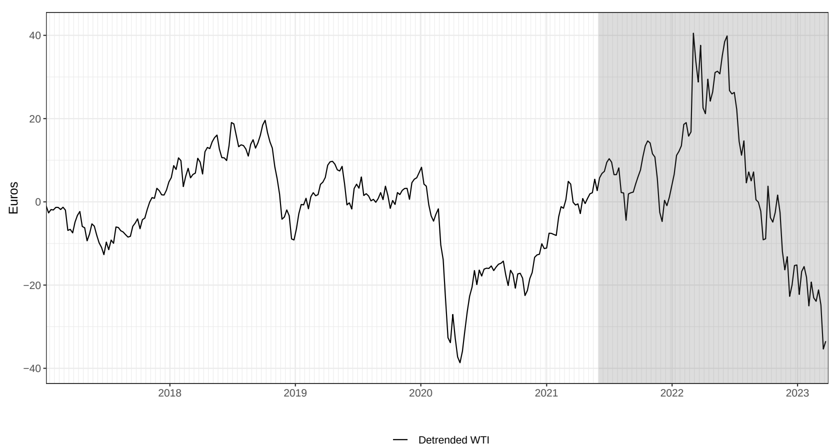
Table 4 reports the estimation results for the MAR(1,1) parameters that drive the dynamics of the bubble assets. The coefficients of the polynomial trend function are significant, supporting the use of the detrending strategy. Most importantly, the non-causal component dominates the causal

[13] As discussed in Hencic and Gouriéroux [2015], it is important to detrend the series by a deterministic function of time and to avoid standard filtering and smoothing procedures that may induce spurious noncausal effects in the data.

Figure 4: Data







Notes: The shaded area corresponds to the out-of-sample data.

one, revealing, for example, the forward-looking steady increase in the oil-price data followed by quite abrupt bubble bursts (see panels B and C). A similar pattern, although with less asymmetry around the shock, is observed in the case of the CO2 series (see panel A). At the same time, Table 5 displays the mean and standard deviation of the bubble-free assets, that are set to follow a GBM as commonly hypothesised in the financial literature.

Based on the in-sample data, we then compute the quantiles and conditional moments of the detrended series. The latter are fed to the CRRA portfolio optimization program with $\gamma \in \{5, 10\}$, which covers two levels of relative risk aversion. Table 6 displays the optimal portfolio strateg(y/ies) in the form of couples (ω, h) identified using the in-sample data for the three portfolios under analysis for a range of quantiles in the upper half of the distribution of the bubble asset. The amount invested, w, may vary quite a lot across quantiles and investors. Apart from some cases that belong to the center of the distribution, the investor has incentives to take short positions. These are generally small in terms of volume of trading (low w) and over relatively short time periods (about one month) independent of the quantile of X_t in the case of the CO2, which corresponds to the particularities of the series. In contrast, the more bubblier oil series are characterized by larger investment horizons, of around 4 months for the Brent and of 6 and a half months for WTI (meaning that very few rebalancings are performed). This finding seems reasonable for investments in oil price, as the largest actors generally take long term hedging positions and the adjustment mechanisms take a long time to be put in place given the strong links with the underlying industry. Besides, for a given quantile, the more risk-averse the investor, the smaller the optimal allocation to the bubble asset.

5.2. Out-of-sample performance analysis

To check the performance of these strategies, we turn to the out-of-sample data. First, we rely on the polynomial coefficient estimates to extend the trend dynamics and remove it from the data.[14] This allows us to match each out-of-sample detrended value that plays the role of a conditioning price, X^d_{T+k}, with the closest floor empirical in-sample quantile of the detrended series and identify the associated portfolio allocation strateg(y/ies). To make the BP approach realistic, we allow for portfolio rebalancing. The investment horizon h being endogenous at each

[14] The polynomial coefficients are reestimated in a recursive framework for each new out-of-sample period, i.e. based on all the previous observed values of the bubble series.

Table 4: Speculative assets: MAR(1,1) and trend estimation

Panel A : CO2					
Detrending method: polynomial of order 4					
(Intercept)	τ^1	τ^2	τ^3	τ^4	
9.0150e+00***	-0.534e+00***	1.541e-02***	-1.143e-04***	2.647e-07***	
α-stable MAR(1,1)					
$\varphi^{\bullet}$	φ°	α	β	σ	μ
0.080***	0.716***	1.625***	-0.103***	0.732***	0.816***
Panel B : Brent					
Detrending method: polynomial of order 4					
(Intercept)	τ^1	τ^2	τ^3	τ^4	
5.441e+01***	-0.413e+00**	01.916e-02***	-1.765e-04***	4.488e-07***	
α-stable MAR(1,1)					
$\varphi^{\bullet}$	φ°	α	β	σ	μ
0.049***	0.908***	1.650***	0.260***	1.480***	1.379***
Panel C : WTI					
Detrending method: polynomial of order 3					
(Intercept)	τ^1	τ^2	τ^3		
3.606e+01***	0.980e+00***	-1.017e-02***	2.772e-05***		
α-stable MAR(1,1)					
$\varphi^{\bullet}$	φ°	α	β	σ	μ
0.158***	0.955***	1.927***	0.944***	1.71***	0.251***

Notes: Estimated parameters of the polynomial trend function and of the α-stable MAR(1,1) process associated with the detrended speculative series for the period 01.2017 - 06.2021. Asterisks *, **, and *** indicate significance at the 90%, 95% and 99% level, respectively.

Table 5: Safer asset returns

EUREX-Euro bund		US 30-year Treasury-bond	
μ	σ	μ	σ
0.0001	0.0077	0.0001	0.0119

Notes: Mean and standard-deviation of the bubble-free asset returns for the period 01.2017 - 06.2021.

Table 6: Optimal portfolio strategies under CRRA

Panel A: CO_2

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 5$)	(0.01,1)	(-0.04,6)	(-0.08,5)	(-0.11,5)	(-0.16,4)	(-0.15,4)	(-0.08,4)	(-0.06,4)	(-0.05,4)
(w^*, h^*)	(0,0)	(0,0)	(0,0)		(-0.13,5)	(-0.12,5)			

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 10$)	(0.01,1)	(-0.02,6)	(-0.04,5)	(-0.06,5)	(-0.07,5)	(-0.08,4)	(-0.04,4)	(-0.03,4)	(-0.02,5)
(w^*, h^*)	(0,0)	(0,0)	(0,0)			(-0.06,5)			

Panel B: Brent

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 5$)	(0.07,1)	(-0.03,23)	(-0.07,20)	(-0.12,17)	(-0.15,15)	(-0.14,15)	(-0.09,14)	(-0.06,15)	(-0.05,15)
(w^*, h^*)	(0.02,17)	(0.03,1)	(0,0)	(0,0)					
	(0,0)	(0,0)							

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 10$)	(0.06,1)	(-0.01,26)	(-0.03,22)	(-0.06,18)	(-0.07,17)	(-0.07,15)	(-0.04,16)	(-0.03,15)	(-0.03,14)
(w^*, h^*)	(0.01,17)	(0.05,1)	(-0.04,19)	(0,0)					(-0.02,18)
	(0,0)	(0,0)	(0.03,1)						
			(0,0)						

Panel C: WTI

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 5$)	(0.02,1)	(-0.02,26)	(-0.06,26)	(-0.13,26)	(-0.22,26)	(-0.27,26)	(-0.22,26)	(-0.09,26)	(-0.09,26)
(w^*, h^*)	(0,0)	(0.02,1)	(0.02,1)						
		(0,0)	(0,0)						

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 10$)	(0.02,1)	(-0.01,26)	(-0.03,26)	(-0.06,26)	(-0.11,26)	(-0.14,26)	(-0.11,26)	(-0.05,26)	(-0.04,26)
(w^*, h^*)	(0,0)	(0.03,1)	(0.04,1)						
		(0,0)	(0,0)						

Notes: The Table displays the optimal portfolio strategies (w^*, h^*) for each portfolio conditional on the quantile of the in-sample distribution in which the bubble asset is at the time of the investment. w is reported in percentages of the investment and h in weeks.

rebalancing step, an irregular, path dependent set of conditioning prices is obtained and the procedure described above to identify the optimal strateg(y/ies) is applied at each time. The position in the bubble asset is closed either when the most recent allocation strategy points to a rebalanced investment horizon h which goes beyond the full investment horizon H, or when a $(\omega^*, h^*) = (0, 0)$ strategy is identified as the optimal. To simplify the understanding of the rebalancing procedure, a decision path is displayed in Figure 5.

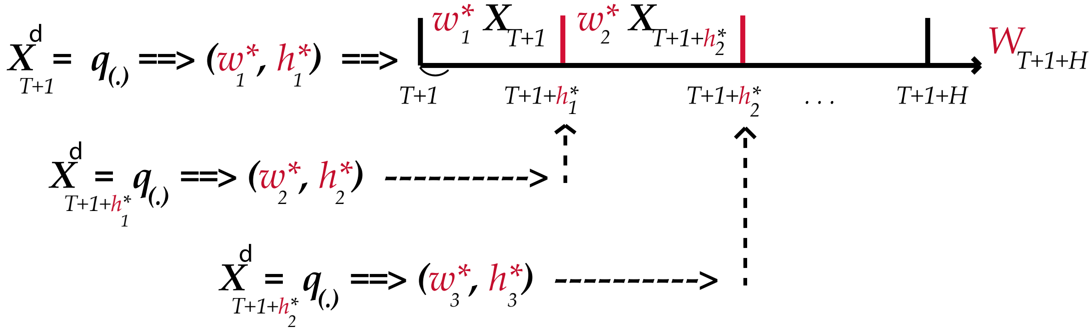


Figure 5: Rebalancing path

As for a given conditioning value an investor can choose among various optimal strategies and (s)he can rebalance the portfolio several times up to the investment horizon H, the performance results for the BP approach for a given initial conditioning price are expressed in terms of quantiles of the distribution of the terminal wealth and other performance measures, and are denoted by $BP_{10\%}$, $BP_{50\%}$, and $BP_{90\%}$, respectively. These results are subsequently aggregated over the out-of-sample (based on the full set of initial conditioning prices).

In Table 7 we report some summary statistics including the average over the out-of-sample of the median and standard deviation of the number of rebalancings and of the number of investment paths. On average, our approach involves a median rebalancing of the portfolio between 2 and 6 weeks over the 6-months horizon H depending of the underlying bubble series. In particular, the CO2-EUREX euro bund portfolio is more often rebalanced than the oil portfolios. This result holds quite uniformly over the distribution of the strategies, i.e. the differences between the three BP strategies are small. The advantage here is that few rebalancings simplify portfolio management operations relative to the daily rebalancing approach of the benchmark portfolios. In contrast, the median of investment paths differs a lot according to the coefficient of relative risk aversion. For example, a more risk averse investor seems to rebalance more, which may quickly lead to a dense tree of possible out-of-sample investment paths over the 26 weeks horizon when we look at all strategies, i.e. $BP_{50\%}$. At the same time, the benchmarks are exogenously set to rebalance every day, as usually done in the literature, and exhibit a unique optimal strategy, which is driven by the weight of the bubble asset in the portfolio. Empirically we observe a quite steady level of these weights over the investment horizon (see the web-appendix for further details). The approach to obtain them is standard in the literature for both benchmarks and shall not be detailed further.

We evaluate the out-of-sample performance of the MAR(1,1)-based strategies through two criteria. First, the out-of-sample terminal wealth, W_{T+H}, associated with each investment path, which is obtained from (1) by applying at each rebalancing date the appropriate optimal strategy (ω, h) to the out-of-sample non-detrended price data. At the same time, we rely on the opportunity cost (see Section 2.3).

The results are reported in Table 8. Each panel (A,B,C) corresponding to a portfolio displays the average and the standard deviation of the various performance measures over the out-of-sample period corresponding to each of the five portfolio strategies under analysis. Asteriks $(*/*)$ associated with the estimated μ of each of our strategies indicate whether the median performance metric for our approach is statistically different from that of the (EW/MV) portfolio according to Wilcoxon's test. To compare the Sharpe and the mSharpe ratios of our approach to those of the benchmark portfolios, we rely on the tests of Ardia and Boudt [2015] with asymptotic HAC standard errors.[15].

The results indicate a positive gain in using our approach relatively to the standard MV and EW portfolios. The terminal wealth indicates a positive return on investment for the BP strategies for all three bubble-asset portfolios and both relative risk aversion coefficients. The only exception is that of the worst 10% strategies of the CO2-EUREX Euro Bund and Brent-EUREX Euro Bund portfolios for the most risk averse investors.

In contrast, the EW strategy is a winning strategy, with a return on investment of about 5%, for the first portfolio and as the degree of non-causality of the speculative asset increases its performance drops (see panels B and C). Note also that there is not much difference in terms

[15]The results are similar when the i.i.d. bootstrap approach is used to compute the standard errors.

Table 7: Number of rebalancings and strategies

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
Panel A: CO2						
CRRA ($\gamma = 5$)	$Med_{N^o Rebalancings}$	6	6	6	26	26
	$\sigma_{N^o Rebalancings}$	0.712	0.622	1.371	0	0
	$Med_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	1.071	0	0	0
CRRA ($\gamma = 10$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$Med_{N^o Rebalancings}$	4.556	4.182	3.952	26	26
	$\sigma_{N^o Rebalancings}$	0.831	0.623	0.518	0	0
	$Med_{N^o Paths}$	15	149	15	1	1
	$\sigma_{N^o Paths}$	47.862	510.238	47.862	0	0
Panel B: Brent						
CRRA ($\gamma = 5$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$Med_{N^o Rebalancings}$	2	2.333	2	26	26
	$\sigma_{N^o Rebalancings}$	0.536	0.173	0.558	0	0
	$Med_{N^o Paths}$	1	3	1	1	1
	$\sigma_{N^o Paths}$	0	0.558	0	0	0
CRRA ($\gamma = 10$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$Med_{N^o Rebalancings}$	2.167	2.4	3	26	26
	$\sigma_{N^o Rebalancings}$	0.674	0.478	0.875	0	0
	$Med_{N^o Paths}$	1	5.5	1	1	1
	$\sigma_{N^o Paths}$	3.717	37.919	3.717	0	0
Panel C: WTI						
CRRA ($\gamma = 5$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$Med_{N^o Rebalancings}$	2	2	2	26	26
	$\sigma_{N^o Rebalancings}$	0.295	0.197	0.635	0	0
	$Med_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.52	0	0	0
CRRA ($\gamma = 10$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$Med_{N^o Rebalancings}$	2	4.182	3.952	26	26
	$\sigma_{N^o Rebalancings}$	0.599	0.623	0.518	0	0
	$Med_{N^o Paths}$	1	149	15	1	1
	$\sigma_{N^o Paths}$	0.612	510.238	47.862	0	0

Notes: The table displays the average over the out-of-sample of the median and the standard deviation of the number of rebalancings and portfolio trajectories in the 10% best (res. worst) performing BP strategies, $BP_{90\%}$ (resp. $BP_{10\%}$) as well as over the full sample of MAR strategies with horizon H ($BP_{50\%}$).

Table 8: Relative performance of portfolio strategies under CRRA

Panel A: CO2							
CRRA ($\gamma = 5$)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	$1.058^{/***}$	$1.069^{/***}$	$1.08^{*/***}$	1.049	0.922
		σ	0.104	0.117	0.127	0.14	0.046
	Sharpe	μ	$0.067^{/**}$	$0.111^{/**}$	$0.127^{/**}$	-0.009	-0.076
		σ	0.509	0.449	0.437	0.684	0.61
	mSharpe	μ	$-0.139^{*/***}$	$-0.129^{/**}$	$-0.102^{/**}$	0.077	-0.128
		σ	0.05	0.064	0.115	0.139	0.088
CRRA ($\gamma = 10$)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	$0.992^{/***}$	$1.14^{***/***}$	$1.361^{***/***}$	1.049	0.922
		σ	0.072	0.104	0.134	0.14	0.046
	Sharpe	μ	$-0.391^{*/}$	$0.023^{**/**}$	$0.354^{**/**}$	-0.009	-0.076
		σ	0.508	0.394	0.495	0.684	0.61
	mSharpe	μ	$-0.115^{**/***}$	$0.028^{**/**}$	$0.169^{***/}$	0.077	-0.128
		σ	0.072	0.101	0.135	0.139	0.088
Panel B: Brent							
CRRA ($\gamma = 5$)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	$1.042^{/***}$	$1.049^{/***}$	$1.072^{**/***}$	1.001	0.921
		σ	0.073	0.079	0.112	0.152	0.044
	Sharpe	μ	$-0.02^{/**}$	$0.091^{*/**}$	$0.17^{**/**}$	-0.031	-0.064
		σ	0.558	0.576	0.562	0.621	0.582
	mSharpe	μ	$-0.151^{*/***}$	$-0.113^{**/**}$	$-0.055^{**/**}$	0.032	-0.128
		σ	0.058	0.073	0.114	0.153	0.077
CRRA ($\gamma = 10$)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	$0.991^{/***}$	$1.085^{***/***}$	$1.234^{***/***}$	1.001	0.921
		σ	0.08	0.1	0.191	0.152	0.044
	Sharpe	μ	$-0.091^{/}$	$0.028^{*/*}$	$0.196^{/}$	-0.031	-0.064
		σ	0.482	0.452	0.423	0.621	0.582
	mSharpe	μ	$-0.125^{***/***}$	$-0.045^{*/*}$	$0.03^{/}$	0.032	-0.128
		σ	0.062	0.101	0.148	0.153	0.077
Panel C: WTI							
CRRA ($\gamma = 5$)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	$1.201^{***/***}$	$1.201^{***/***}$	$1.201^{***/***}$	0.99	0.932
		σ	0.177	0.176	0.176	0.167	0.056
	Sharpe	μ	$0.016^{/*}$	$0.111^{/*}$	$0.122^{/}$	0.012	-0.024
		σ	0.59	0.421	0.42	0.611	0.608
	mSharpe	μ	$-0.111^{/}$	$-0.107^{/*}$	$-0.078^{/*}$	0.043	-0.084
		σ	0.085	0.085	0.149	0.195	0.082
CRRA ($\gamma = 10$)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	$1.145^{***/***}$	$1.227^{***/***}$	$1.312^{***/***}$	0.99	0.932
		σ	0.189	0.177	0.179	0.167	0.056
	Sharpe	μ	$-0.173^{/*}$	$0.033^{/}$	$0.21^{/}$	0.012	-0.024
		σ	0.569	0.485	0.412	0.611	0.608
	mSharpe	μ	$-0.105^{/*}$	$-0.004^{/}$	$0.079^{/}$	0.043	-0.084
		σ	0.078	0.158	0.219	0.195	0.082

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth, Sharpe ratio and modified Sharpe ratio. The results take the form of out-of-sample average and standard deviation. Wilkoxon's test is used for the terminal wealth and we rely on the tests by Ardia and Boudt [2015] for the (m)Sharpe ratios. Asteriks ($*/*$) indicate the rejection of the null hypothesis of each test at the 90%, 95% and 99% levels.

Table 9: Relative performance of portfolio strategies under CRRA: opportunity cost

			EW vs			MV vs	
Panel A: CO2							
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.009	0.02	0.031	0.137	0.147	0.158
	σ	0.211	0.22	0.228	0.127	0.138	0.149
CRRA ($\gamma = 10$)	μ	-0.056	0.092	0.312	0.071	0.219	0.44
	σ	0.151	0.155	0.179	0.084	0.114	0.141
Panel B: Brent							
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.041	0.048	0.071	0.121	0.128	0.151
	σ	0.217	0.221	0.242	0.099	0.104	0.134
CRRA ($\gamma = 10$)	μ	-0.01	0.084	0.233	0.07	0.164	0.313
	σ	0.188	0.201	0.295	0.105	0.126	0.215
Panel C: WTI							
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.21	0.21	0.211	0.269	0.269	0.269
	σ	0.315	0.314	0.314	0.22	0.22	0.22
CRRA ($\gamma = 10$)	μ	0.154	0.236	0.322	0.213	0.295	0.38
	σ	0.287	0.249	0.247	0.206	0.187	0.193

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of opportunity cost (OC).

of wealth dispersion between our approach and the *EW*, although one notices an increase in the standard deviation with the degree of non-causality. Most strikingly, the mean-variance approach always exhibits the lowest dispersion and a negative return on investment. This result suggests that the *MV* strategy is not fit for bubble assets.

The (m)Sharpe ratios support the previous findings, as in most cases they are the highest for the *BP* strategies. Besides, the comparison tests generally indicate a significant difference in the average performance which is in favor of our approach.

Finally, Table 9 reports the opportunity cost relative performance measure for the three portfolios under analysis in two cases, without and with transaction costs. Recall that a positive value indicates the amount that must be added to the return of the benchmark strategy, such that it leaves the investor indifferent to the decision between it and the corresponding *BP* strategy. The OC is generally positive, hence supporting the superiority of our strategy relative to the benchmarks. The only case where the *EW* portfolio is preferred is when the investor bets on the 10% worst *BP* strategies for CO2 and Brent bubble assets.

All in all, the MAR(1,1) process captures well the bubble behaviour of the series and this is shown to generally materialize in better portfolio performance relative to the traditional benchmarks.

5.3. Robustness Analysis

We now investigate the sensitivity of our results to various changes in the portfolio analysis by: i) accounting for transaction costs, ii) relying on a different optimisation algorithm, the PGP, and iii) by ignoring the higher-order moments in the portfolio optimisation.

First, transaction costs play an important role in the investment decisions as they impact portfolio performance proportionally to the number and / or value of rebalancings. For this reason, we gauge their impact on the performance of our portfolio strategies relative to the benchmarks in a simple setup where transaction costs are fixed at 0.05% per unit of investment. The results, available in the Web-Appendix for space-contraints are qualitatively similar to those without transaction costs. Note however that the *BP* strategies register a slightly larger drop in terminal wealth than the benchmarks: 0.010 vs 0.003 on average. Indeed, as the transaction costs are proportional to the shares of both assets that are exchanged, although our approach involves sparse rebalancing, the amounts involved in each trade are larger than those rebalanced daily by the benchmark portfolios (see Figure 1 in the Web-Appendix for the dynamics of EW and MV portfolio weights).

At the same time, we rely on the PGP as an alternative portfolio allocation approach that explicitly accounts for higher-order moments in asset price distribution. For technical details on PGP see Appendix B. As in the CRRA case, we identify the optimal portfolio strategies under PGP based on the in-sample data. Table 10 reports these strateg(y/ies) for a range of quantiles in the upper half of the distribution of each of the three bubble assets. Notice that when the conditioning price is in the quantiles close to the center of the distribution, the investor takes almost no position as the weights are either 0 or very close to it. In contrast, when the investment takes place further in the bubble period, two types of comparable optimal – according to the PGP criterion defined – strategies arise: a long and a short one. The shorter one is characterized by a larger investment share in the bubble asset over a possibly slightly shorter horizon (down by one to five weeks depending on the portfolio and the sensitivity of the investor to kurtosis). The investor who weights more the fourth moment appears to choose more extreme allocations in the bubble asset for both long and short strategies. Another main difference with respect to the CRRA case is that the PGP investor does not take the risk to wait until the terminal horizon and see whether the bubble continues and hence does not short the positions beyond 19 trading weeks for the 0.99 quantile and above. Still, the more bubblier oil series are characterized by larger investment horizons than the CO2, as in the CRRA case.

To gauge the performance of our MAR(1,1)-based approach using the optimal strategies under PGP, we proceed similarly to the case of the CRRA utility function. Table 11 reports the summary statistics including the average over the out-of-sample of the median and standard deviation of the number of rebalancings and of the number of investment paths. Similarly to the CRRA case, the CO2-EUREX Euro bund portfolio is still the most often rebalanced, while the WTI-US T-bond is the least rebalanced one with small variations from one quantile to another. Note also that the most dense tree of investment paths under PGP corresponds to the CO2-EUREX Euro bund portfolio irrespective of the sensitivity of the investor to the kurtosis. This is most certainly due to the fact that the optimal investment horizon h is at most equal to 2 while in the upper quantiles the investor can chose each time between a long and a short strategy.

We evaluate the out-of-sample performance of the BP strategies under PGP through the criteria discussed in section 2.3.[16] The results are reported in Table 12.[17] The distribution of the wealth seems to be more spread than in the CRRA case, with a $BP_{90\%}$ strategy similar to that in Table 8 but with a much inferior $BP_{10\%}$ strategy.

Still, the results indicate a positive gain in using our approach relatively to the standard MV and EW portfolios. Indeed, the return on investment is positive in all three panels for the median and upper quantiles of the BP approach and both PGP specifications. In contrast, the EW strategy beats $BP_{10\%}$. The (m)Sharpe ratios support the CRRA findings and the comparison tests are most often in favor of our approach. Results are also robust to the presence of transaction costs, with the same caveats as in the CRRA case (see the web-appendix).

Finally, we gauge the optimal portfolio allocation according to the BP strategies when the higher-order moments are ignored. In the spirit of a stress-test, this analysis puts our MAR(1,1)-based approach on a more equal footing with the benchmarks, which make use only of information in the first two moments.

The main insight here is that ... For space-constraints all Tables and detailed analyses are defered to the web-appendix.

5.4. Tree path analysis

To better illustrate an investor's decision-making process, we consider the case of the WTI - US Treasury-bond portfolio and choose the out-of-sample starting point so as to be in a bubble period. More precisely, the investment time is set to 04 March 2022, when the WTI detrended price is in the 80% quantile of its in-sample distribution.

Figure 6 displays the possible investment paths for the two optimisation algorithms and the different investor preferences used in the main analysis. At each rebalancing time, the optimal strateg(y/ies) are expressed as $\text{sign}(w) * h.w$. 0 denotes the initial investment time and 27 the closure of all positions once the investment horizon H is reached.

Panel (a) reveals that in the CRRA (γ=5) case there is only one path, involving a single portfolio rebalancing which occurs the day before closing the positions. A similar path would follow a more risk averse investor, who, in turn, would invest a shorter amount in the bubble asset

[16]Since the PGP is not a utility-based allocation approach, we cannot compute the opportunity cost relative performance measure.

[17]As the PGP approach is not utility-based, we cannot rely on the opportunity cost as a performance measure anymore.

Table 10: Optimal portfolio strategies under PGP

Panel A: CO_2

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP (1,1,1,1) (w^*, h^*)	(-0.01,1) (0,0)	(-0.01,1) (0,0)	(-0.01,2) (0,0)	(-0.01,2) (0,0)	(-0.01,2) (0,0)	(-0.02,1) (0,0)	(-0.66,1) (0.32,2) (0,0)	(-0.54,1) (0.29,2) (0,0)	(-0.5,1) (0.29,2) (0,0)

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP(1,1,1,4) (w^*, h^*)	(0,0)	(0,0)	(0,0)	(-0.01,1) (0,0)	(-0.01,2) (0,0)	(-0.01,2) (0,0)	(-0.95,1) (0.57,1) (0,0)	(-0.79,1) (0.56,1)	(-0.73,1) (0.52,1)

Panel B: Brent

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP (1,1,1,1) (w^*, h^*)	(-0.01,2) (0.01,2) (0,0)	(-0.02,2) (-0.01,4) (0,0)	(-0.02,3) (-0.01,6) (0,0)	(-0.02,4) (-0.01,7) (0,0)	(-0.03,3) (-0.01,7) (0,0)	(-0.04,3) (0,0)	(-0.76,3) (0.38,6)	(-0.58,3) (0.31,6)	(-0.53,3) (0.28,6)

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP(1,1,1,4) (w^*, h^*)	(0,0)	(0,0)	(-0.01,6) (0,0)	(-0.01,7) (0,0)	(-0.01,7) (0,0)	(-0.01,7) (0,0)	(-0.95,4) (0.63,5)	(-0.73,4) (0.51,5)	(-0.66,4) (0.46,5)

Panel C: WTI

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP (1,1,1,1)	(-0.02,3) (0.01,3) (0,0)	(-0.03,4) (0,0)	(-0.04,4) (0,0)	(-0.05,5) (0,0)	(-0.05,6) (0,0)	(-0.06,6) (-0.01,20) (0,0)	(-0.11,6) (-0.01,17) (0,0)	(-0.71,6) (0.38,11)	(-0.67,6) (0.36,11)

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP(1,1,1,4)	(0.01,21) (-0.01,18) (-0.02,12) (0,0)	(-0.02,14) (-0.01,20) (0.01,22) (0.02,14) (0,0)	(-0.02,15) (-0.03,12) (-0.01,22) (0.01,21) (0.02,14) (0,0)	(-0.03,13) (-0.02,16) (-0.01,23) (0,0)	(-0.03,13) (-0.02,16) (-0.01,24) (0,0)	(-0.03,12) (-0.02,16) (-0.01,23) (0,0)	(-0.06,9) (-0.01,19) (0,0)	(0.63,8) (-0.94,8)	(0.6,8) (-0.88,8)

Notes: The table displays the optimal portfolio strategies for each portfolio conditional on the quantile of the in-sample distribution in which the bubble asset is at the time of the investment. w is reported in percentages of the investment and h in weeks.

Table 11: Number of rebalancings and strategies

Panel A: CO2

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	$Med_{N^o Rebalancings}$	7	6.662	7.125	26	26
	$\sigma_{N^o Rebalancings}$	3.928	3.788	4.191	0	0
	$Med_{N^o Paths}$	8	76	8	1	1
	$\sigma_{N^o Paths}$	275.071	2750.47	275.071	0	0
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	$Med_{N^o Rebalancings}$	6	6.375	6.2	26	26
	$\sigma_{N^o Rebalancings}$	3.382	3.341	3.343	0	0
	$Med_{N^o Paths}$	4	40	4	1	1
	$\sigma_{N^o Paths}$	120.032	1200.957	120.032	0	0

Panel B: Brent

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	$Med_{N^o Rebalancings}$	7	6.549	7.063	26	26
	$\sigma_{N^o Rebalancings}$	1.341	1.156	1.382	0	0
	$Med_{N^o Paths}$	9	85	9	1	1
	$\sigma_{N^o Paths}$	2.689	27.673	2.689	0	0
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	$Med_{N^o Rebalancings}$	6	6.098	6.2	26	26
	$\sigma_{N^o Rebalancings}$	2.05	1.981	2.139	0	0
	$Med_{N^o Paths}$	3	29.5	3	1	1
	$\sigma_{N^o Paths}$	1.517	17.33	1.517	0	0

Panel C: WTI

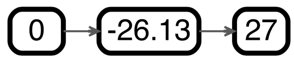
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	$Med_{N^o Rebalancings}$	3	2.75	3	26	26
	$\sigma_{N^o Rebalancings}$	0.978	0.337	1.138	0	0
	$Med_{N^o Paths}$	1	7	1	1	1
	$\sigma_{N^o Paths}$	0.308	2.758	0.308	0	0
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	$Med_{N^o Rebalancings}$	2	1.667	1	26	26
	$\sigma_{N^o Rebalancings}$	0.43	0.24	0.732	0	0
	$Med_{N^o Paths}$	1	3	1	1	1
	$\sigma_{N^o Paths}$	0	1.057	0	0	0

Notes: The table displays the average over the out-of-sample of the median and the standard deviation of the number of rebalancings and optimal portfolio trajectories in the 10% best ($BP_{90\%}$) and worst ($BP_{10\%}$) performing BP strategies as well as over the full sample of BP strategies ($BP_{50\%}$) with horizon H.

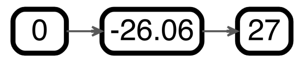
Table 12: Relative performance of portfolio strategies under PGP

Panel A: CO2							
PGP (1,1,1,1)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	0.992$^{/***}$	1.061$^{*/***}$	1.19$^{***/***}$	1.049	0.922
		σ	0.075	0.078	0.178	0.14	0.046
	Sharpe	μ	-0.37$^{/}$	0.094$^{/}$	0.704$^{/*}$	-0.009	-0.076
		σ	1.157	0.891	1.201	0.684	0.61
	mSharpe	μ	-0.165$^{/*}$	-0.058$^{/*}$	0.095$^{*/}$	0.077	-0.128
		σ	0.103	0.124	0.198	0.139	0.088
PGP (1,1,1,4)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	0.976$^{**/***}$	1.049$^{/***}$	1.149$^{***/***}$	1.049	0.922
		σ	0.086	0.076	0.133	0.14	0.046
	Sharpe	μ	-0.671$^{/}$	0.089$^{/}$	0.745$^{/}$	-0.009	-0.076
		σ	1.291	0.548	1.144	0.684	0.61
	mSharpe	μ	-0.13$^{/}$	-0.003$^{/}$	0.213$^{/}$	0.077	-0.128
		σ	0.083	0.131	0.257	0.139	0.088
Panel B: Brent							
PGP (1,1,1,1)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	0.978$^{/***}$	1.055$^{*/***}$	1.203$^{***/***}$	1.001	0.921
		σ	0.09	0.08	0.159	0.152	0.044
	Sharpe	μ	-0.204$^{*/*}$	0.091$^{*/**}$	0.38$^{*/*}$	-0.031	-0.064
		σ	0.435	0.493	0.528	0.621	0.582
	mSharpe	μ	-0.188$^{/**}$	-0.1$^{*/**}$	0.046$^{*/*}$	0.032	-0.128
		σ	0.054	0.057	0.145	0.153	0.077
PGP (1,1,1,4)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	0.961$^{*/**}$	1.05$^{*/***}$	1.193$^{***/***}$	1.001	0.921
		σ	0.099	0.087	0.151	0.152	0.044
	Sharpe	μ	-0.266$^{/}$	0.082$^{/}$	0.405$^{/}$	-0.031	-0.064
		σ	0.497	0.574	0.672	0.621	0.582
	mSharpe	μ	-0.165$^{/}$	-0.07$^{/*}$	0.101$^{/}$	0.032	-0.128
		σ	0.071	0.074	0.154	0.153	0.077
Panel C: WTI							
PGP (1,1,1,1)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	0.983$^{/**}$	1.061$^{**/***}$	1.137$^{***/***}$	0.99	0.932
		σ	0.168	0.094	0.155	0.167	0.056
	Sharpe	μ	-0.249$^{/}$	0.076$^{/}$	0.422$^{/}$	0.012	-0.024
		σ	0.578	0.482	0.72	0.611	0.608
	mSharpe	μ	-0.158$^{**/***}$	-0.094$^{/}$	0.038$^{/}$	0.043	-0.084
		σ	0.06	0.08	0.17	0.195	0.082
PGP (1,1,1,4)			$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	Wealth	μ	1.065$^{***/***}$	1.068$^{***/***}$	1.08$^{***/***}$	0.99	0.932
		σ	0.081	0.081	0.098	0.167	0.056
	Sharpe	μ	-0.139$^{/}$	0.06$^{/}$	0.246$^{/}$	0.012	-0.024
		σ	0.329	0.396	0.393	0.611	0.608
	mSharpe	μ	-0.129$^{***/***}$	-0.08$^{/}$	-0.009$^{/}$	0.043	-0.084
		σ	0.065	0.058	0.038	0.195	0.082

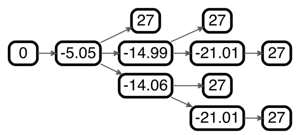
Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth, Sharpe ratio and modified Sharpe ratio. The results take the form of out-of-sample average and standard deviation. Wilkoxon's test is used for the terminal wealth and we rely on the tests by Ardia and Boudt [2015] for the (m)Sharpe ratios. Asteriks ($*/*$) indicate the rejection of the null hypothesis of each test at the 90%, 95% and 99% levels.



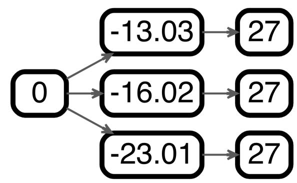
(a) CRRA (γ=5)



(b) CRRA (γ=10)



(c) PGP(1,1,1,1)



(d) PGP(1,1,1,4)

Figure 6: Decision-making trees
Reading key: sign$(w) * h.w$, with w the investment share in the bubble asset and h the horizon of this investment.

(w=0.06 in panel (b) instead of 0.13 in the previous case). In the PGP case the tree is more dense, with five possible paths when the four conditional moments of the return distribution are equally important and three such paths when the investor pays more attention to the kurtosis.

6. Conclusion

In this paper we propose an asset allocation strategy particularly designed for the case of an investors wishing to include a bubble asset in his/her portfolio. For this, we account explicitly for the distributional characteristics of bubble assets through a MAR(1,1) model, which seems to be appropriate to capture locally explosive behaviours. The higher-order conditional moments of the return distribution are then plugged in the Taylor-series-expansion of the CRRA utility function and the PGP algorithm, respectively. The economic value of the *BP* strategy is compared in out-of-sample with standard benchmarks such as the mean-variance and equally-weighted portfolios based on well-known performance measures such as the opportunity cost, the Sharpe ratio and the modified Sharpe ratio. Both simulation-based analyses and empirical results using CO2, Brent and WTI data for the bubble asset support the superiority of our approach. The stronger the anticipative character of the bubble behaviour of the asset, i.e. the larger the non-causal parameter, the wider the hedging possibilities and the more substantial the possible gains involved.

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Appendix A. Proofs

Appendix A.1. Conditional moments of returns

For the bubble asset we have the non-central moments of returns

$$\mathbb{E}(r_{t,t+h}^{Xj}|X_t) = \frac{1}{X_t}\mathbb{E}(X_{t,t+h}^{p}|X_t), \quad \text{for} \quad p \geq 1,$$

where the conditional moments of the price series are defined through Proposition 1.

For the second asset, using the properties of the GBM, one obtains

$$
\begin{aligned}
\mathbb{E}(r^{S}_{t,t+h}|X_t) &= e^{\upsilon h} - 1 \\
\mathbb{E}(r^{S^2}_{t,t+h}|X_t) &= e^{2\upsilon h+\varsigma^2 h} - 2e^{\upsilon h} + 1 \\
\mathbb{E}(r^{S^3}_{t,t+h}|X_t) &= e^{3\upsilon h+3\varsigma^2 h} - 3e^{2\upsilon h+\varsigma^2 h} + 3e^{\upsilon h} - 1 \\
\mathbb{E}(r^{S^4}_{t,t+h}|X_t) &= e^{4\upsilon h+4\varsigma^2 h} - 4e^{3\upsilon h+3\varsigma^2 h} + 6e^{2\upsilon h+\varsigma^2 h} - 4e^{\upsilon h} + 1,
\end{aligned}
$$

and similar expressions, using $H-h$ instead of h, can be derived for $\mathbb{E}(r^{S}_{t+h,t+H}|X_t)$. Making use of the mapping relations between central and non-central moments and the independence between the two assets, one can subsequently express $\mathbb{E}\left[(R_{t+H} - \overline{R}_{t+H})^k | X_t, S_t\right]$ as a function of these moments.

Appendix A.2. Derivation of the constants σ_1^{α}, β_1, κ_p, and λ_p

Fries 2021 shows that if X_t is a α-stable two-sided MA(∞) process with $0 < \alpha < 2, \alpha \neq 1$, $\beta \in [-1,1]$, and $\sigma > 0$ as defined in Section 2.2, i.e. well defined, stationary process with α-stable errors, and for $h \geq 1$ then one can obtain the conditional moments of the process X_t for $p \leq 4$ with

$$
\sigma_1 = \sigma^{\alpha} \sum_{k\in\mathbb{Z}} |a_k|^{\alpha}, \qquad \beta_1 = \beta \frac{\sum_{k\in\mathbb{Z}} a_k^{<\alpha>}}{\sum_{k\in\mathbb{Z}} |a_k|^{\alpha}}, \qquad \kappa_p = \frac{\sum_{k\in\mathbb{Z}} |a_k|^{\alpha} (\frac{a_{k-h}}{a_k})^p}{\sum_{k\in\mathbb{Z}} |a_k|^{\alpha}}, \qquad \lambda_p = \frac{\sum_{k\in\mathbb{Z}} a_k^{<\alpha>} (\frac{a_{k-h}}{a_k})^p}{\sum_{k\in\mathbb{Z}} |a_k|^{\alpha}},
$$

where $y^{<\alpha>} = \text{sign}(y)|y|^{\alpha}$ for any $y \in \mathbb{R}$. Using his results together with the fact that the coefficients of the $MA(\infty)$ representation of a MAR(1,1) process, $X_t = \sum_{k=-\infty}^{\infty} a_k \varepsilon_{t-k}$, satisfy

$$
\begin{aligned}
a_k &= \frac{\varphi^{\circ k}}{1 - \varphi^{\circ}\varphi^{\bullet}} \qquad \text{if } k \geq 0, \\
a_k &= \frac{\varphi^{\bullet -k}}{1 - \varphi^{\circ}\varphi^{\bullet}} \qquad \text{otherwise,}
\end{aligned}
$$

and calculus based on geometric series, one can easily obtain the results in Proposition 1.

Appendix B. PGP optimisation framework

As discussed in the introduction, the polynomial goal programming is an alternative portfolio allocation approach to the CRRA. We follow Aksarayah and Pala (2018) to implement the PGP model based on the first four conditional moments of the return distribution. This approach is not subjected to a Taylor approximation (as the CRRA utility function case), so the weights it attributes to the portfolio moments cannot be precisely related to the parameters of a utility function. In this case, the wealth W_t of the investor is allocated at time t by solving conflicted multi objectives such as maximizing expected return and skewness and minimizing variance and kurtosis that are weighted with investor preferences. The PGP model can be defined as

$$\min_{(\omega,h)} \Big(1+|d_1-R^*|\Big)^{\gamma_1} + \Big(1+|d_2-V^*|\Big)^{\gamma_2} + \Big(1+|d_3-S^*|\Big)^{\gamma_3} + \Big(1+|d_4-K^*|\Big)^{\gamma_4}, \tag{B.1}$$

$$\text{s.t. } R_{(\omega,h)} + d1 = R^*, \;\; V_{(\omega,h)} - d2 = V^*, \;\; S_{(\omega,h)} + d3 = S^*, \;\; K_{(\omega,h)} - d4 = K^*, \;\; d_i \geq 0,$$

where $R_{(\omega,h)}$, $V_{(\omega,h)}$, $S_{(\omega,h)}$, and $K_{(\omega,h)}$ denote respectively the expectation, variance, skewness and excess kurtosis of the returns R_{t+H} conditional on the price level $X_t = x$ for a given strategy (ω, h) defined as

$$R_{\omega,h} := \mathbb{E}\Big[R_{t+H}|X_t, S_t\Big], \;\; V_{\omega,h} := \mathbb{E}\Big[(R_{t+H} - R_{\omega,h})^2|X_t, S_t\Big],$$
$$S_{\omega,h} := \mathbb{E}\Big[(R_{t+H} - R_{\omega,h})^3/V_{\omega,h}^{3/2}|X_t, S_t\Big], \;\; K_{\omega,h} := \mathbb{E}\Big[(R_{t+H} - R_{\omega,h})^4/V_{\omega,h}^2|X_t, S_t\Big] - 3,$$

Besides, R^*, V^*, S^*, K^* denote the optima of the subprograms $\max_{(\omega,h)} R_{(\omega,h)}$, $\min_{(\omega,h)} V_{(\omega,h)}$, $\max_{(\omega,h)} S_{(\omega,h)}$, $\min_{(\omega,h)} K_{(\omega,h)}$, and the γ_i's are non-negative parameters weighting the preference of the investor to pursue optimality of one moment over the others. All these necessary quantities are obtained from the conditional predictive distribution of the MAR(1,1) process associated with the bubble asset and the GBM dynamics of the bubble-free asset, as previously. To check the sensitivity of the results to the investor's appraisal of higher moments, we consider two cases: one in which the investor allocates the same weight to the four moments, i.e. γ_1=γ_2=γ_3=γ_4=1, hereafter denoted PGP(1,1,1,1), and one in which the investor pays more attention to the temperance (kurtosis), i.e. γ_1=γ_2=γ_3=1, and γ_4=4, hereafter denoted PGP(1,1,1,4).

Appendix : Bet on a bubble asset ? An optimal portfolio allocation strategy

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May 2023
(work in progress, do not cite)

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1. Monte-Carlo Simulations

This section includes additional simulation results on the economic value of the proposed portfolio allocation strategy, which are consistent with those in the mail paper. The BP strategies in Tables 1 to 6 in Subsection 1.2 are based on the first four moments of the return distribution of the bubble asset, while the BP2 strategies in Tables to in Subsection 1.2 rely only on the first two moments. The average of the median terminal wealth for the three MAR(1,1)-based portfolios is in most cases well above that of the benchmark portfolios and superior to the invested amount. The only exception is the worse BP strategy, which in the PGP case is slightly inferior to 1. The positive averages of the opportunity cost also support these findings. Moreover, the performance of the BP2 strategies is dominated by that of BP, which is consistent with the DGP. Note also that in the BP2 case there is only one PGP strategy, as the fourth moment, which differentiated then before, is now irrelevant.

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1.1. *BP strategies based on the first four moments (BP)*

Table 1: CRRA ($\gamma = 5$): Relative performance of portfolio strategies

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW	
Wealth	μ	$1.018^{*/*}$	1.005	$1.032^{*/*}$	1.003	1.002	
	σ	0.048	0.048	0.046	0.052	0.006	
			EW vs			MV vs	
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.011	-0.003	0.043	0.010	0.001	0.014
	σ	0.062	0.062	0.063	0.046	0.046	0.047

Notes: Our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth, Sharpe ratio and modified Sharpe ratio. The opportunity cost (OC) relatively to the two benchmark portfolios is also provided. The results take the form of out-of-sample average and standard deviation over the 1000 simulations. Asteriks $(*/*)$ indicate the rejection of the null hypothesis of Wilcoxon's test of equality of medians at the 95% level relatively to each of the two benchmark strategies, EW and MV, respectively.

Table 2: CRRA ($\gamma = 5$): Relative performance of portfolio strategies in positive bubble period

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV	
Wealth	μ	$1.067^{*/*}$	$1.065^{*/*}$	$1.069^{*/*}$	1.031	1.019	
	σ	0.021	0.021	0.021	0.030	0.020	
			vs EW			vs MV	
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.025	0.025	0.025	0.012	0. 012	0.019
	σ	0.055	0.020	0.020	0.050	0.020	0.020

Notes: see note to Table 1. The results are based only on the cases where the investment is performed while the first asset exhibits a bubble period, i.e. $X_{T+k} = x$ is beyond the 95% quantile of the theoretical distribution of the price process.

Table 3: PGP(1,1,1,1): Relative performance of portfolio strategies

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW
Wealth	μ	$1.021^{*/*}$	0.995	$1.095^{*/*}$	1.003	1.002
	σ	0.093	0.017	0.019	0.052	0.006

Notes: see note to Table 1.

Table 4: PGP(1,1,1,1): Relative performance of portfolio strategies in positive bubble period

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV
Wealth	μ	$1.036^{*/*}$	$0.999^{*/*}$	$1.055^{*/*}$	1.031	1.019
	σ	0.038	0.008	0.011	0.030	0.020

Notes: see note to Table 2.

Table 5: PGP(1,1,1,4): Relative performance of portfolio strategies

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW
Wealth	μ	$1.022^{*/*}$	0.996	$1.101^{*/*}$	1.003	1.002
	σ	0.007	0.019	0.020	0.052	0.006

Notes: see note to Table 1.

Table 6: PGP(1,1,1,4): Relative performance of portfolio strategies in positive bubble period

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV
Wealth	μ	$1.044^{*/*}$	$0.999^{*/*}$	$1.071^{*/*}$	1.031	1.019
	σ	0.021	0.011	0.012	0.030	0.020

Notes: see note to Table 2.

1.2. *BP strategies based on the first two moments (BP2)*

Table 7: CRRA ($\gamma = 10$): Relative performance of BP2 portfolio strategies

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW	
Wealth	μ	$1.018^{*/*}$	1.006	$1.034^{*/*}$	1.005	1.001	
	σ	0.108	0.179	0.180	0.109	0.006	
		EW vs			MV vs		
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.013	0.005	0.027	0.016	0.011	0.028
	σ	0.112	0.131	0.133	0.096	0.135	0.177

Notes: Our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth, Sharpe ratio and modified Sharpe ratio. The opportunity cost (OC) relatively to the two benchmark portfolios is also provided. The results take the form of out-of-sample average and standard deviation over the 1000 simulations. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilcoxon's test of equality of medians at the 95% level relatively to each of the two benchmark strategies, EW and MV, respectively.

Table 8: CRRA ($\gamma = 10$): Relative performance of BP2 portfolio strategies in positive bubble period

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV	
Wealth	μ	$1.020^{*/*}$	$1.002^{*/*}$	$1.023^{*/*}$	1.005	1.001	
	σ	0.025	0.025	0.026	0.108	0.006	
		vs EW			vs MV		
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.034	0.031	0.036	0.039	0.032	0.046
	σ	0.099	0.099	0.010	0.025	0.026	0.025

Notes: see note to Table 7. The results are based only on the cases where the investment is performed while the first asset exhibits a bubble period, i.e. $X_{T+k} = x$ is beyond the 95% quantile of the theoretical distribution of the price process.

Table 9: CRRA ($\gamma = 5$): Relative performance of BP2 portfolio strategies

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW
Wealth	μ	$1.018^{*/*}$	1.006	$1.034^{*/*}$	1.005	1.001
	σ	0.108	0.179	0.180	0.109	0.006

		EW vs			MV vs		
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.022	0.018	0.027	0.024	0.016	0.031
	σ	0.149	0.149	0.150	0.168	0.169	0.168

Notes: see note to Table 7.

Table 10: CRRA ($\gamma = 5$): Relative performance of BP2 portfolio strategies in positive bubble period

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV
Wealth	μ	$1.028^{*/*}$	$1.028^{*/*}$	$1.028^{*/*}$	1.005	1.001
	σ	0.026	0.026	0.026	0.108	0.006

		vs EW			vs MV		
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.031	0.031	0.030	0.092	0.003	0.092
	σ	0.100	0.100	0.010	0.026	0.026	0.026

Notes: see note to Table 8.

Table 11: PGP : Relative performance of BP2 portfolio strategies

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW
Wealth	μ	$1.017^{*/*}$	0.986	$1.048^{*/*}$	1.005	1.001
	σ	0.067	0.078	0.078	0.109	0.006

Notes: see note to Table 7.

Table 12: PGP : Relative performance of BP2 portfolio strategies in positive bubble period

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV
Wealth	μ	$1.019^{*/*}$	$1.000^{*/*}$	$1.033^{*/*}$	1.005	1.001
	σ	0.018	0.011	0.030	0.108	0.006

Notes: see note to Table 8.

2. Empirical Application

This section includes supplementary results on portfolio allocation with a bubble asset, where the latter is either the CO2, the Brent or the WTI price. More precisely, in Subsection 2.1 we discuss the case where one accounts for transaction costs, much relevant for any investor, and in Subsection 2.2 the case where the BP optimisation exploits only the information contained in the first two conditional moments, which puts our approach on a more equal footing with the benchmarks.

2.1. Transaction costs

We gauge the impact of transaction costs on the performance of our portfolio strategies relative to the benchmarks in a simple setup where transaction costs are fixed at 0.05% per unit of investment. The results for the CRRA approach are displayed in Tables 13 and 14, while those for PGP are reported in Table 15. They support the main findings of the paper, in particular the superiority of the BP approach over the two benchmarks.

The BP strategies generally register a slightly larger drop in terminal wealth than the benchmarks: 0.010 vs 0.003 on average. This is specific to our setup of transaction costs that are proportional to the investment in each asset. Indeed, although our approach involves sparse rebalancing, the quantities exchanged in each trade are larger than the sum of those rebalanced daily by the benchmark portfolios. This can be easily seen if one compares w in Tables 6 and 11 of the paper with the dynamics of the weights for the benchmarks displayed in Figure 1 below. The latter depicts the weights of the benchmark models obtained within a one-period-ahead rolling window scheme that follows that of the rebalancing. Their slow-varying behaviour suggests very low ajustements of the portfolio and thus very low proportional transaction costs.

However, even in presence of transaction costs the results indicate a positive gain in using the BP approach relative to the two benchmarks. It is only in the case of the 10% worst strategies that the opportunity cost is negative and the *EW* portfolio is preferred.

Figure 1: EW and MV weights

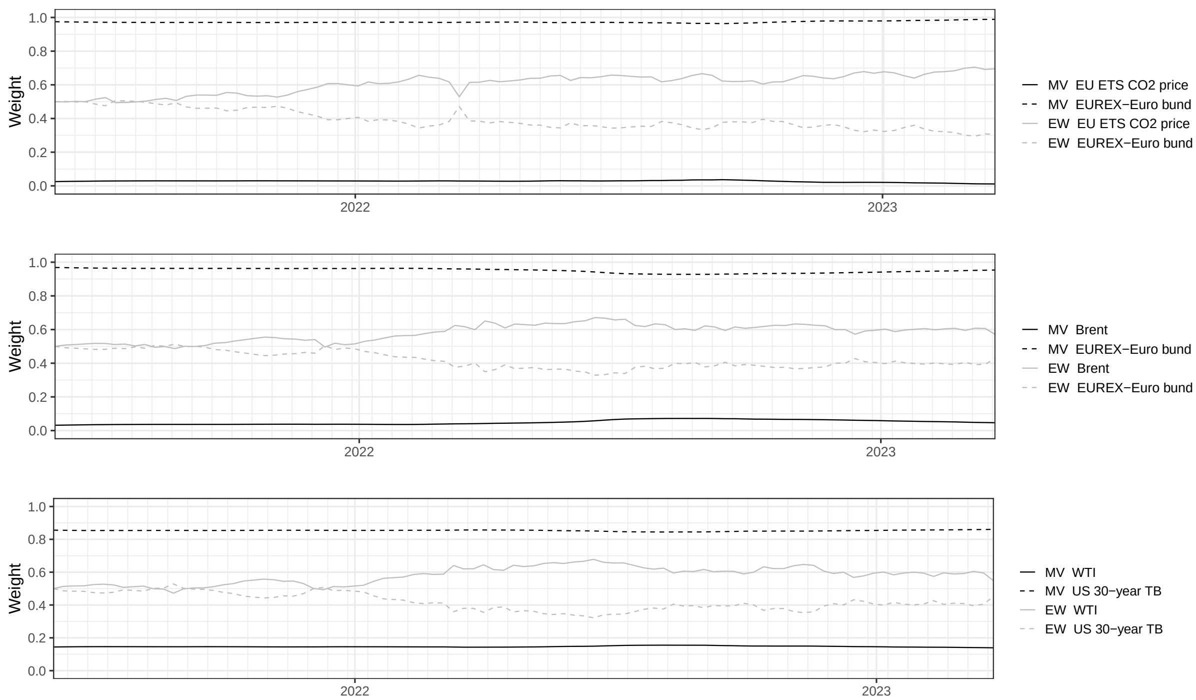

Table 13: Relative performance of portfolio strategies under CRRA with transaction costs

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.048^{/***}$	$1.059^{/***}$	$1.070^{*/***}$	1.045	0.922
	σ	0.103	0.115	0.125	0.139	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$0.984^{/***}$	$1.130^{***/***}$	$1.345^{***/***}$	1.045	0.922
	σ	0.071	0.103	0.135	0.139	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.032^{/***}$	$1.038^{/***}$	$1.061^{**/***}$	0.998	0.921
	σ	0.073	0.078	0.111	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$0.983^{/***}$	$1.074^{***/***}$	$1.221^{***/***}$	0.998	0.921
	σ	0.080	0.099	0.188	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.190^{***/***}$	$1.190^{***/***}$	$1.191^{***/***}$	0.987	0.932
	σ	0.176	0.176	0.175	0.167	0.055
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.135^{***/***}$	$1.216^{***/***}$	$1.302^{***/***}$	0.987	0.932
	σ	0.188	0.176	0.177	0.167	0.055

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth. The results take the form of out-of-sample average and standard deviation. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilkoxon's test at the 90%, 95% and 99% levels. The transaction costs are fixed at 0.05% per unit of investment.

Table 14: Relative performance of portfolio strategies under CRRA: opportunity cost (with transaction costs)

Panel A: CO2

		EW vs			MV vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.124	0.013	0.024	0.126	0.137	0.148
	σ	0.110	0.218	0.226	0.125	0.136	0.147
CRRA ($\gamma = 10$)	μ	-0.124	0.085	0.300	0.062	0.208	0.424
	σ	0.110	0.155	0.179	0.083	0.113	0.142

Panel B: Brent

		EW vs			MV vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.077	0.041	0.063	0.111	0.118	0.140
	σ	0.142	0.220	0.240	0.100	0.104	0.132
CRRA ($\gamma = 10$)	μ	-0.077	0.076	0.223	0.062	0.153	0.300
	σ	0.142	0.201	0.292	0.105	0.126	0.212

Panel C: WTI

		EW vs			MV vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.055	0.203	0.203	0.258	0.258	0.259
	σ	0.132	0.313	0.313	0.219	0.219	0.218
CRRA ($\gamma = 10$)	μ	-0.055	0.229	0.315	0.203	0.284	0.370
	σ	0.132	0.248	0.246	0.205	0.186	0.191

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (*EW*) and mean-variance (*MV*) ones in terms of opportunity cost (OC). Transaction costs are fixed at 0.05% per unit of investment.

Table 15: Relative performance of portfolio strategies under PGP with transaction costs

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.984^{/***}$	$1.051^{*/***}$	$1.177^{***/***}$	1.045	0.922
	σ	0.075	0.077	0.174	0.139	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$0.968^{**/***}$	$1.041^{/***}$	$1.139^{***/***}$	1.045	0.922
	σ	0.086	0.075	0.130	0.139	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.968^{/***}$	$1.045^{*/***}$	$1.19^{***/***}$	0.998	0.921
	σ	0.090	0.079	0.155	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$0.952^{*/**}$	$1.041^{*/***}$	$1.181^{***/***}$	0.998	0.921
	σ	0.100	0.087	0.147	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.974^{/**}$	$1.052^{**/***}$	$1.127^{***/***}$	0.987	0.932
	σ	0.168	0.094	0.152	0.167	0.055
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$1.056^{***/***}$	$1.059^{***/***}$	$1.072^{***/***}$	0.987	0.932
	σ	0.081	0.080	0.096	0.167	0.055

Notes: see note to Table 13. The transaction costs are fixed at 0.05% per unit of investment.

2.2. First two conditional moments

This setup allows one to study the performance of the bubble portfolio strategies (labeled BP2) when only the first two conditional moments are taken into account for each of the three assets. The bubble-asset dynamics is still assumed to follow a MAR(1,1) process whose estimated parameters are reported in Table 4 of the paper. However, the CRRA utility maximization program in equation (2) and the PGP model in equation (B.1) rely only on the first two conditional moments of the distribution of the terminal wealth in this setup. By comparison with the results of the main paper, the findings here are expected to shed light on the contribution of third and fourth conditional moments to the performance of the bubble portfolio strategies.

The resulting optimal portfolio strateg(y/ies) in the form of couples (ω, h) are reported in Tables 16 and 22, respectively, while the number of rebalancings are displayed in Tables 17 and 23. They highlight shorter positions on longer horizons and less rebalancing than in the BP case.

Finally, the performance results are reported in Tables 18 - 21, 24 and 25. Note that in the BP2 case there is only one PGP strategy, as the fourth moment, which differentiated then before, is now irrelevant. The performance of BP2 strategies seems to be slightly better than that of BP for the CO2 data, but the results are more mitigated for the other two bubble series. We therefore suggest one to rely on the more general four-moments based approach and use BP2 for sensitivity analysis.

Table 16: Optimal BP2 portfolio strategies under CRRA

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
Panel A: CO2									
CRRA ($\gamma = 5$)	(0.01,1)	(-0.04,7)	(-0.1,7)	(-0.19,7)	(-1,7)	(-1,8)	(-0.16,7)	(-0.1,7)	(-1,8)
(w^*, h^*)	(0,0)	(0,0)	(0,0)		(-0.41,7)		(-1,8)	(-1,8)	(-0.09,7)
CRRA ($\gamma = 10$)	(0.01,1)	(-0.02,8)	(-0.05,7)	(-0.09,7)	(-1,9)	(-1,10)	(-1,10)	(-1,11)	(-1,11)
(w^*, h^*)	(0,0)	(0,0)	(0,0)						
Panel B: Brent									
CRRA ($\gamma = 5$)	(0.07,1)	(-0.03,26)	(-0.1,24)	(-0.19,24)	(-0.33,24)	(-1,25)	(-1,26)	(-1,26)	(-1,26)
(w^*, h^*)	(0.02,18)	(0.03,1)	(0,0)			(-0.42,24)	(-0.19,24)	(-0.11,24)	
	(0,0)	(0,0)							
CRRA ($\gamma = 10$)	(0.06,1)	(-0.01,26)	(-0.05,25)	(-0.09,25)	(-1,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)
(w^*, h^*)	(0.01,18)	(-0.02,24)	(0.02,1)						
	(0,0)	(0.05,1)	(0,0)						
		(0,0)							
Panel C: WTI									
CRRA ($\gamma = 5$)	(0.02,1)	(-0.02,26)	(-0.08,26)	(-0.22,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)
(w^*, h^*)	(0.01,26)	(0.02,1)	(0.02,1)						
	(0,0)	(0,0)	(0,0)						
CRRA ($\gamma = 10$)	(0.02,1)	(-0.01,26)	(-0.04,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)
(w^*, h^*)	(0,0)	(0.03,1)	(0.04,1)						
		(0,0)							

Notes: The Table displays the optimal portfolio strategies (w^*, h^*) for each portfolio conditional on the quantile of the in-sample distribution in which the bubble asset is at the time of the investment. w is reported in percentages of the investment and h in weeks.

Table 17: Number of rebalancings and BP2 strategies under CRRA

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
Panel A: CO2						
CRRA ($\gamma = 5$)	$\mu_{N^o Rebalancings}$	3.000	3.000	3.000	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.611	0.471	0.642	0	0
	$\mu_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.653	0	0	0
CRRA ($\gamma = 10$)	$\mu_{N^o Rebalancings}$	3.462	3.333	3.333	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.813	0.572	0.461	0	0
	$\mu_{N^o Paths}$	10	101	10	1	1
	$\sigma_{N^o Paths}$	17.206	181.978	17.215	0	0
Panel B: Brent						
CRRA ($\gamma = 5$)	$\mu_{N^o Rebalancings}$	2.000	2 .000	2.000	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.173	0.085	0.121	0	0
	$\mu_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.398	0	0	0
CRRA ($\gamma = 10$)	$\mu_{N^o Rebalancings}$	2.000	2 .000	2.000	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.405	0.364	0.642	0	0
	$\mu_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	2.817	29.92	2.817	0	0
Panel C: WTI						
CRRA ($\gamma = 5$)	$\mu_{N^o Rebalancings}$	2.000	2.000	2.000	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.171	0.149	0.606	0	0
	$\mu_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.813	0	0	0
CRRA ($\gamma = 10$)	$\mu_{N^o Rebalancings}$	2.000	3.333	3.333	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.593	0.572	0.461	0	0
	$\mu_{N^o Paths}$	1	101	10	1	1
	$\sigma_{N^o Paths}$	0.708	181.978	17.215	0	0

Notes: The table displays the average over the out-of-sample of the median and the standard deviation of the number of rebalancings and portfolio trajectories in the 10% best (res. worst) performing BP strategies, $BP_{90\%}$ (resp. $BP_{10\%}$) as well as over the full sample of MAR strategies with horizon H ($BP_{50\%}$).

Table 18: Relative performance of BP2 portfolio strategies under CRRA (without transaction costs)

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$0.999^{*/***}$	$1.007^{/***}$	$1.026^{/***}$	1.049	0.922
	σ	0.234	0.236	0.247	0.140	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$0.939^{***/***}$	$1.116^{***/***}$	$1.346^{***/***}$	1.049	0.922
	σ	0.122	0.119	0.132	0.140	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.192^{***/***}$	$1.192^{***/***}$	$1.194^{***/***}$	1.001	0.921
	σ	0.175	0.174	0.173	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.108^{***/***}$	$1.154^{***/***}$	$1.230^{***/***}$	1.001	0.921
	σ	0.166	0.157	0.189	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.247^{***/***}$	$1.248^{***/***}$	$1.253^{***/***}$	0.990	0.932
	σ	0.194	0.193	0.189	0.167	0.056
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.146^{***/***}$	$1.246^{***/***}$	$1.325^{***/***}$	0.990	0.932
	σ	0.188	0.186	0.190	0.167	0.056

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth. The results take the form of out-of-sample average and standard deviation. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilkoxon's test at the 90%, 95% and 99% levels.

Table 19: Relative performance of BP2 portfolio strategies under CRRA (with transaction costs)

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$0.988^{*/***}$	$0.996^{/***}$	$1.015^{/***}$	1.045	0.922
	σ	0.231	0.234	0.244	0.139	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$0.928^{***/***}$	$1.099^{***/***}$	$1.326^{***/***}$	1.045	0.922
	σ	0.122	0.113	0.131	0.139	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.181^{***/***}$	$1.182^{***/***}$	$1.183^{***/***}$	0.998	0.921
	σ	0.174	0.174	0.172	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.098^{***/***}$	$1.143^{***/***}$	$1.218^{***/***}$	0.998	0.921
	σ	0.165	0.156	0.187	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.236^{***/***}$	$1.237^{***/***}$	$1.242^{***/***}$	0.987	0.932
	σ	0.193	0.192	0.188	0.167	0.055
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.136^{***/***}$	$1.235^{***/***}$	$1.314^{***/***}$	0.987	0.932
	σ	0.187	0.185	0.188	0.167	0.055

Notes: see note to Table 18 The transaction costs are fixed at 0.05% per unit of investment.

Table 20: Relative performance of BP2 portfolio strategies under CRRA: opportunity cost

Panel A: CO2

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.050	-0.042	-0.023	0.077	0.085	0.104
	σ	0.323	0.325	0.334	0.249	0.251	0.262
CRRA ($\gamma = 10$)	μ	-0.110	0.067	0.297	0.017	0.194	0.424
	σ	0.201	0.180	0.167	0.134	0.131	0.132

Panel B: Brent

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.191	0.191	0.193	0.271	0.271	0.273
	σ	0.302	0.302	0.302	0.204	0.204	0.203
CRRA ($\gamma = 10$)	μ	0.107	0.153	0.229	0.187	0.233	0.309
	σ	0.258	0.256	0.294	0.191	0.185	0.217

Panel C: WTI

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.257	0.257	0.263	0.315	0.316	0.321
	σ	0.336	0.335	0.336	0.238	0.237	0.236
CRRA ($\gamma = 10$)	μ	0.156	0.255	0.335	0.214	0.314	0.394
	σ	0.285	0.239	0.250	0.205	0.190	0.202

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (*EW*) and mean-variance (*MV*) ones in terms of opportunity cost (OC).

Table 21: Relative performance of BP2 portfolio strategies under CRRA: opportunity cost (with transaction costs)

Panel A: CO2

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.124	-0.049	-0.003	0.067	0.074	0.093
	σ	0.110	0.322	0.331	0.246	0.248	0.259
CRRA ($\gamma = 10$)	μ	-0.124	0.054	0.281	0.007	0.178	0.404
	σ	0.110	0.175	0.163	0.134	0.123	0.130

Panel B: Brent

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.077	0.184	0.186	0.261	0.261	0.263
	σ	0.142	0.301	0.301	0.203	0.203	0.202
CRRA ($\gamma = 10$)	μ	-0.077	0.145	0.220	0.177	0.222	0.297
	σ	0.142	0.255	0.292	0.190	0.184	0.215

Panel C: WTI

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.191	0.191	0.193	0.271	0.271	0.273
	σ	0.302	0.302	0.302	0.204	0.204	0.203
CRRA ($\gamma = 10$)	μ	0.107	0.153	0.229	0.187	0.233	0.309
	σ	0.258	0.256	0.294	0.191	0.185	0.217

Notes: see note to Table 20. Transaction costs are fixed at 0.05% per unit of investment.

Table 22: Optimal BP2 portfolio strategies under PGP

Panel A: CO2

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP	(-0.01,7)	(-0.11,7)	(-0.23,7)	(-0.38,7)	(-0.55,7)	(-0.54,7)	(-0.28,7)	(-0.18,7)	(-0.17,7)
(w^*, h^*)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Panel B: Brent

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP	(0.04,23)	(-0.09,23)	(-0.23,24)	(-0.4,23)	(-0.55,23)	(-0.54,23)	(-0.31,24)	(-0.2,23)	(-0.18,24)
(w^*, h^*)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Panel C: WTI

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP	(0.01,26)	(-0.06,26)	(-0.18,26)	(-0.37,26)	(-0.69,26)	(-0.95,26)	(-0.97,26)	(-0.37,26)	(-0.35,26)
	(0,0)	(0,0)	(0,0)	(0,0)				(0,0)	(0,0)

Notes: The Table displays the optimal portfolio strategies (w^*, h^*) for each portfolio conditional on the quantile of the in-sample distribution in which the bubble asset is at the time of the investment. w is reported in percentages of the investment and h in weeks.

Table 23: Number of rebalancings and BP2 strategies under PGP

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
Panel A: CO2						
PGP (1,1,1,1)	$Med_{N^o Rebalancings}$	4	4	4	67	67
	$\sigma_{N^o Rebalancings}$	0	0	0	0	0
	$Med_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0	0	0	0
PGP (1,1,1,4)	$Med_{N^o Rebalancings}$	4	4	4	67	67
	$\sigma_{N^o Rebalancings}$	0	0	0	0	0
	$Med_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0	0	0	0
Panel B: Brent						
PGP (1,1,1,1)	$Med_{N^o Rebalancings}$	2	1.5	1	68	68
	$\sigma_{N^o Rebalancings}$	0.477	0	0.477	0	0
	$Med_{N^o Paths}$	1	2	1	1	1
	$\sigma_{N^o Paths}$	0	0	0	0	0
PGP (1,1,1,4)	$Med_{N^o Rebalancings}$	2	1.5	1	68	68
	$\sigma_{N^o Rebalancings}$	0.477	0	0.477	0	0
	$Med_{N^o Paths}$	1	2	1	1	1
	$\sigma_{N^o Paths}$	0	0	0	0	0
Panel C: WTI						
PGP (1,1,1,1)	$Med_{N^o Rebalancings}$	2	2	2	67	67
	$\sigma_{N^o Rebalancings}$	0.386	0.215	0.239	0	0
	$Med_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.43	0	0	0
PGP (1,1,1,4)	$Med_{N^o Rebalancings}$	2	2	2	67	67
	$\sigma_{N^o Rebalancings}$	0.386	0.215	0.239	0	0
	$Med_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.43	0	0	0

Notes: The table displays the average over the out-of-sample of the median and the standard deviation of the number of rebalancings and portfolio trajectories in the 10% best (res. worst) performing BP strategies, $BP_{90\%}$ (resp. $BP_{10\%}$) as well as over the full sample of MAR strategies with horizon H ($BP_{50\%}$).

Table 24: Relative performance of BP2 portfolio strategies under PGP (without transaction costs)

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.024^{/***}$	$1.024^{/***}$	$1.024^{/***}$	1.049	0.922
	σ	0.088	0.088	0.088	0.140	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.024^{/***}$	$1.038^{/***}$	$1.052^{*/***}$	1.001	0.921
	σ	0.090	0.081	0.074	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.229^{***/***}$	$1.240^{***/***}$	$1.252^{***/***}$	0.990	0.932
	σ	0.191	0.187	0.188	0.167	0.056

Notes: The quantiles of our MAR(1,1)-based strategies with first two moments are compared with the equally weighted (*EW*) and mean-variance (*MV*) ones in terms of terminal wealth. The results take the form of out-of-sample average and standard deviation. Asteriks $(*/*)$ indicate the rejection of the null hypothesis of Wilkoxon's test at the 90%, 95% and 99% levels.

Table 25: Relative performance of BP2 portfolio strategies under PGP (with transaction costs)

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.014^{/***}$	$1.014^{/***}$	$1.014^{/***}$	1.045	0.922
	σ	0.088	0.088	0.088	0.139	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.016^{/***}$	$1.030^{/***}$	$1.045^{*/***}$	0.998	0.921
	σ	0.091	0.081	0.072	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.219^{***/***}$	$1.230^{***/***}$	$1.241^{***/***}$	0.987	0.932
	σ	0.189	0.186	0.187	0.167	0.055

Notes: see note to Table 24. The transaction costs are fixed at 0.05% per unit of investment.