Web-Appendix: Bet on a bubble asset ? An optimal portfolio allocation strategy

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This Web-Appendix is designed to provide complementary results with respect to those included in the manuscript. It is organized as follows. Section 1 describes the PGP optimisation framework. Section 2 investigates the impact of parameter estimation on portfolio allocation in a Monte-Carlo experiment, while Section 3 includes additional simulation results on the economic value of the proposed portfolio allocation strategy relatively to Section 3 in the main paper. Finally, Section 4 provides a variety of robustness checks with respect to the empirical application discusses in the main paper.

1 PGP optimisation framework

As discussed in the introduction of the main paper, the polynomial goal programming is an alternative portfolio allocation approach to the CRRA. We follow Aksarayah and Pala (2018) to implement the PGP model based on the first four conditional moments of the return distribution. This approach is not subjected to a Taylor approximation (as the CRRA utility function case), so the weights it attributes to the portfolio moments cannot be precisely related to the parameters of a utility function. In this case, the wealth W_t of the investor is allocated at time t by solving conflicted multi objectives such as maximizing expected return and skewness and minimizing variance and kurtosis that are weighted with investor preferences. The PGP model can be defined as

$$\min_{(\omega,h)} \Big(1+|d_1-R^*|\Big)^{\gamma_1}+\Big(1+|d_2-V^*|\Big)^{\gamma_2}+\Big(1+|d_3-S^*|\Big)^{\gamma_3}+\Big(1+|d_4-K^*|\Big)^{\gamma_4}, \quad (1)$$

$$\text{s.t. } R_{(\omega,h)}+d1=R^*, \quad V_{(\omega,h)}-d2=V^*, \quad S_{(\omega,h)}+d3=S^*, \quad K_{(\omega,h)}-d4=K^*, \quad d_i\geq 0,$$

where $R_{(\omega,h)}$, $V_{(\omega,h)}$, $S_{(\omega,h)}$, and $K_{(\omega,h)}$ denote respectively the expectation, variance, skewness and excess kurtosis of the returns R_{t+H} conditional on the price level $X_t = x$ for a given strategy (ω, h) defined as

$$R_{\omega,h} := \mathbb{E}\Big[R_{t+H}|X_t, S_t\Big], \quad V_{\omega,h} := \mathbb{E}\Big[(R_{t+H} - R_{\omega,h})^2|X_t, S_t\Big],$$
$$S_{\omega,h} := \mathbb{E}\Big[(R_{t+H} - R_{\omega,h})^3/V_{\omega,h}^{3/2}|X_t, S_t\Big], \quad K_{\omega,h} := \mathbb{E}\Big[(R_{t+H} - R_{\omega,h})^4/V_{\omega,h}^2|X_t, S_t\Big] - 3,$$

Besides, R^*, V^*, S^*, K^* denote the optima of the subprograms $\max_{(\omega,h)} R_{(\omega,h)}$, $\min_{(\omega,h)} V_{(\omega,h)}$, $\max_{(\omega,h)} S_{(\omega,h)}$, $\min_{(\omega,h)} K_{(\omega,h)}$, and the γ_i's are non-negative parameters weighting the preference of the investor to pursue optimality of one moment over the others. All these necessary quantities are obtained from the conditional predictive distribution of the MAR(1,1) process associated with the bubble asset and the GBM dynamics of the bubble-free asset, as previously. To check the sensitivity of the results to the investor's appraisal of higher moments, we consider two cases: one in which the investor allocates the same weight to the four moments, i.e. $\gamma_1=\gamma_2=\gamma_3=\gamma_4=1$, hereafter denoted PGP(1,1,1,1), and one in which the investor pays more attention to the temperance (kurtosis), i.e. $\gamma_1=\gamma_2=\gamma_3=1$, and $\gamma_4=4$, hereafter denoted PGP(1,1,1,4).

2 Monte-Carlo Simulations

This section is designed to check the reliability of our approach to design optimal portfolio strategies. As the investor does not have perfect knowledge of the parameters of the distribution of the speculative asset, we investigate the impact of parameter estimation on portfolio allocation in a Monte-Carlo experiment. We adopt a parametric plug-in estimation approach and proceed in two steps.[1] First, we gauge the sensitivity of the conditional moments of returns to parameter estimation and then we look into the variability this induces in the optimal portfolio strategy.

[1] A model-free non-parametric approach could also be envisaged, but it would engender a dramatic loss in efficiency, especially for conditioning values $X_t = x$ far away from the central values of the process (X_t), (see Fries, 2021, Supplementary Material).

We simulate $M = 2000$ trajectories of $N = \{250, 1000, 5000\}$ observations from the same MAR(1,1) process as in Section 3 of the paper: $(1 - 0.9F)(1 - 0.1B)X_t = \varepsilon_t$ where $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.3, 1.5, 15)$[2]. We then estimate the conditional power moments by replacing the theoretical constants $\sigma_1^\alpha,\ \beta_1,\ \kappa_p,\ \lambda_p$. in Proposition 1 in the main paper by their empirical counterparts computed by plugging-in the MAR(1,1) parameter estimates obtained by Maximum Likelihood.[3]

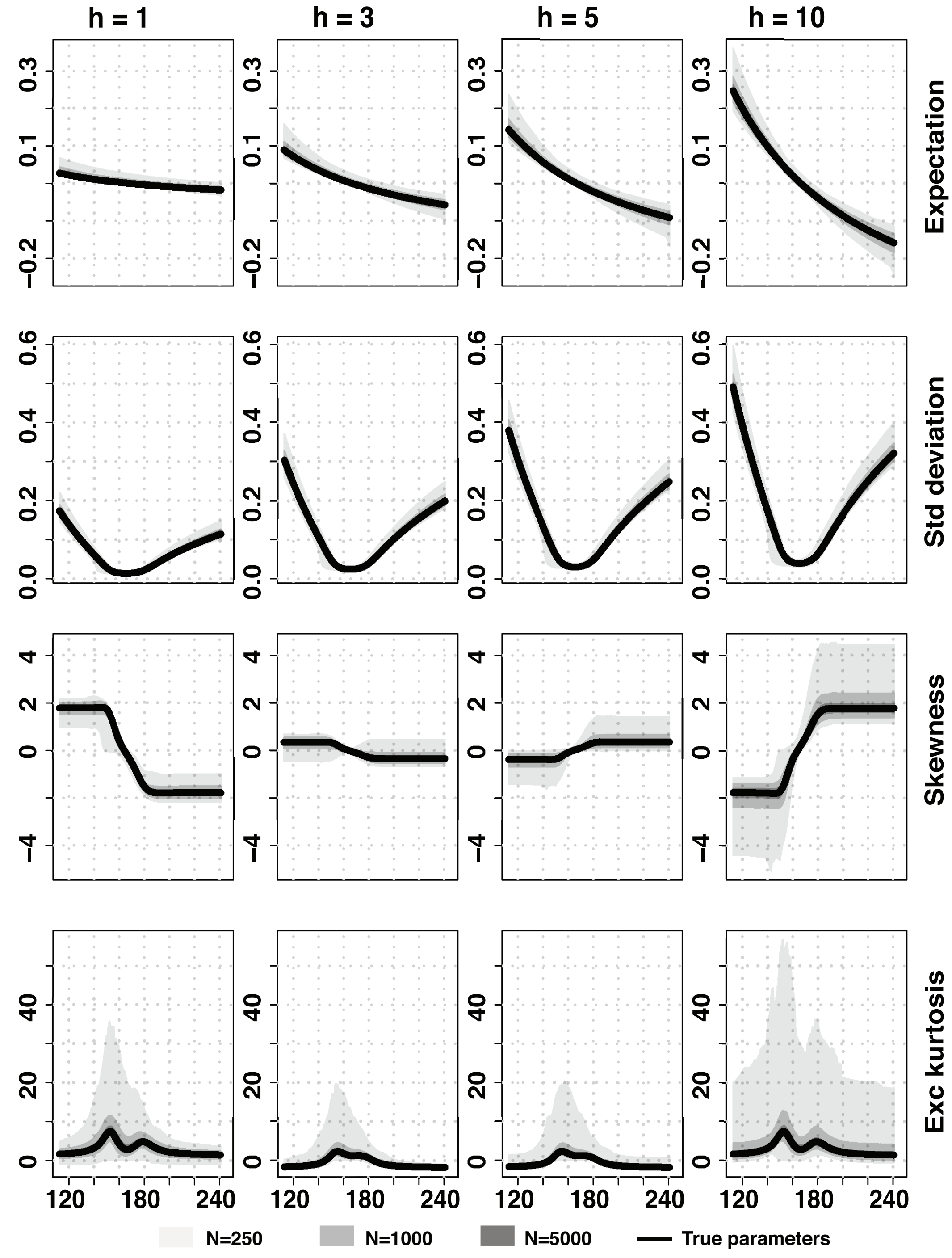
The results are displayed in Figure 1 for prediction horizons $h = 1, 3, 5, 10$ and conditioning values $x \in (112, 245)$ that correspond to the 0.05% and 99.95% quantiles of the marginal distribution of X_t. These results take the form of a pointwise 5% - 95% interquartile interval of the conditional moment estimators for each sample size N. Notice that the theoretical conditional moments, based on the true values of the parameters and represented by a black line, always belong to the empirical interquartile range. More precisely, the interquantile intervals are narrow around most of the true conditional moments curves, even for small sample sizes. They are larger for higher-order moments and large horizons when $N = 250$ but narrow down fast as the sample size increases. Overall, the plug-in method appears to be a good way to estimate the conditional moments even when the conditioning values $X_t = x$ are in the tails of the marginal distribution of the process.

In the second step we hence investigate the impact of parameter estimation on the selected portfolios. The simulated conditional moments of returns obtained from the ML estimates of the MAR(1,1) process are plugged in the CRRA portfolio optimization program to get the optimal portfolio strategi(es) in the form of couples (ω^*, h^*), which define the part of the wealth to invest in the bubble asset and the horizon of this investment given that the overall investment horizon is fixed to $H = 26$ periods, i.e., six-months of weekly

[2]This choice of parameters makes the process satisfy the rational bubble condition, i.e. $\left[(\varphi^\circ)^{\alpha-1} + \varphi^\bullet(1 - (\varphi^\circ)^\alpha)\right]^{-1} < 1$, (see Remark 4.1 in Fries, 2021)

[3]To facilitate the estimation, we initialize the parameters of the α stable distribution by relying on the approach of McCulloch (1986). Provided the ML estimator is consistent, which is the case for the one used here (see Andrews et al., 2009), the plug-in estimators of the conditional moments will also be consistent.

Figure 1: Conditional moments of return distribution



Notes: First four conditional moments of returns when the price series follows a MAR(1,1) process $(1 - 0.9F)(1 - 0.13B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.3, 1.5, 15)$. The conditional moments are obtained for conditioning values $X_t = x \in (112, 245)$, i.e. 99.9% of the probability mass of the marginal distribution of X_t is supported on this interval. The black line represents the theoretical moments, whereas the gray shaded areas correspond to the simulated conditional moments based on 2000 draws from the above α-stable distribution for the three sample sizes. In the latter case the MAR(1,1) parameters are estimated by Maximum Likelihood and plugged in the formulae of Proposition 1. The results are displayed for three horizons, $h = 1, 3, 5, 10$ and three sample sizes, $N = 250, 1000, 5000$.

trading activity. To be more precise, we search for optima (ω^*, h^*) in the set $[-1, 1] \times [0, 26]$, thus allowing short strategies. As the optimization program is likely non-convex, several strategies may lead to the same terminal wealth, and in this case they are all labeled as optimal strategies. We round ω^* to the closest percentage point and report h^* in weeks. Besides, by convention, if either $\omega^* = 0$ or $h^* = 0$, we report $(\omega^*, h^*) = (0, 0)$. For the bubble-free asset, we set υ and ς so that the annual return and volatility equal 2%.

Figures 2 and 3 propose a visualization of the optimal investment strategies if the DGP were known and of the impact of parameter estimation on the selected optimal portfolios for a CRRA investor with risk aversion parameter $\gamma = 10$. For each starting value of the speculative asset $X_t = x$ defined by a specific quantile of its distribution and each sample size N, we plot the mass repartition of the estimated strategies across the 2000 simulations in the share-horizon space. The bigger and redder the dots, the larger the mass of portfolios falling in that area. Roughly speaking, a red circle corresponds to more than 1000 identical strategies, a violet one indicate more than 500 identical ones, whereas the smallest blue dots represent between 5 and 50 identical strategies.[4] The optimal strategies under the hypothesis that the investor knows the parameters of the speculative asset dynamics are denoted by black target symbols.

While the starting value of S_t does not matter, the starting value of X_t deeply modifies the investment landscape. The first figure looks into the case of conditioning values in the lower conditional quantiles of X_t. The CRRA investor bets on a rising value of the speculative asset and opts for a full investment in it ($\omega^* = 1$) over horizons of up to fifteen weeks ahead, $h^* < 15$. This long strategy is the only optimal portfolio allocation in this setup, i.e., the equilibrium is unique for X_t outside the trough (0.0001 quantile). The optima from the simulated strategies, denoted by colored dots, are generally concentrated in the vicinity of the true optimal strategies, which indicates that estimation uncertainty

[4] We do not report the precise values as they vary from one subplot to another due to the multiple equilibrium issue discussed earlier and the plot would become too dense to be easily readable.

does not affect much the portfolio allocation problem. As the sample size N increases, the estimation becomes even more accurate and more mass gathers around the true optima.

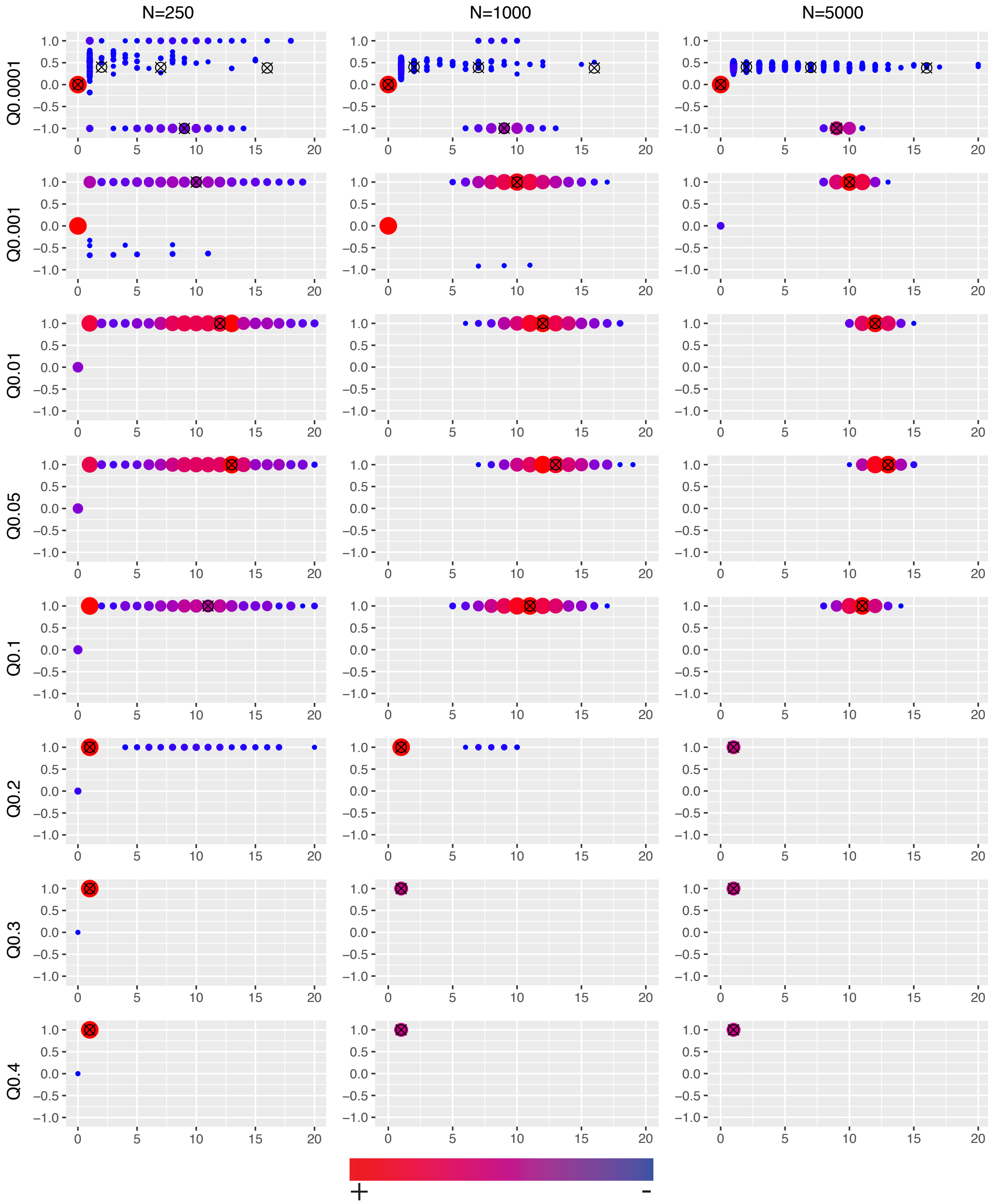
The second figure depicts the case of conditioning values at the median and in the upper conditional quantiles of X_t. The long strategy, characterized by a share close to 1 invested over very short intervals, is optimal as we move from the center of the distribution towards the bubble zone.

Multiple optimal strategies arise as we get to the steepest part of the inflation phase of the bubble. This comes in hand with different investors betting on different scenarios according to their risk adversity. Shorting the bubble over a 2-period horizon appears to be as optimal as investing a certain share of wealth over a horizon between 2 and 7 periods. Next, in the explosive regime of the 0.99 quantile the optimal strategy is to completely short the bubble asset over a one-period horizon. Finally, above the 0.99 quantile, i.e. as the explosive regime becomes more evident, the optimal strategy consists in a fair short position over 12 to 14 periods, which is consistent with the increasing bubble crash risk.

Additionally, the dispersion of simulation-based strategies around the true ones rapidly shrinks with the sample size, suggesting that the true optima can indeed be consistently retrieved after parameter estimation. For the quantiles furthest in the tail, the dispersion is in the horizon dimension rather than in the share dimension. Estimation uncertainty on the verge of a bubble crash thus mainly impacts the holding horizon. The results are robust to the choice of the risk-aversion parameter and to changes in the speculative asset price data generating process.[5]

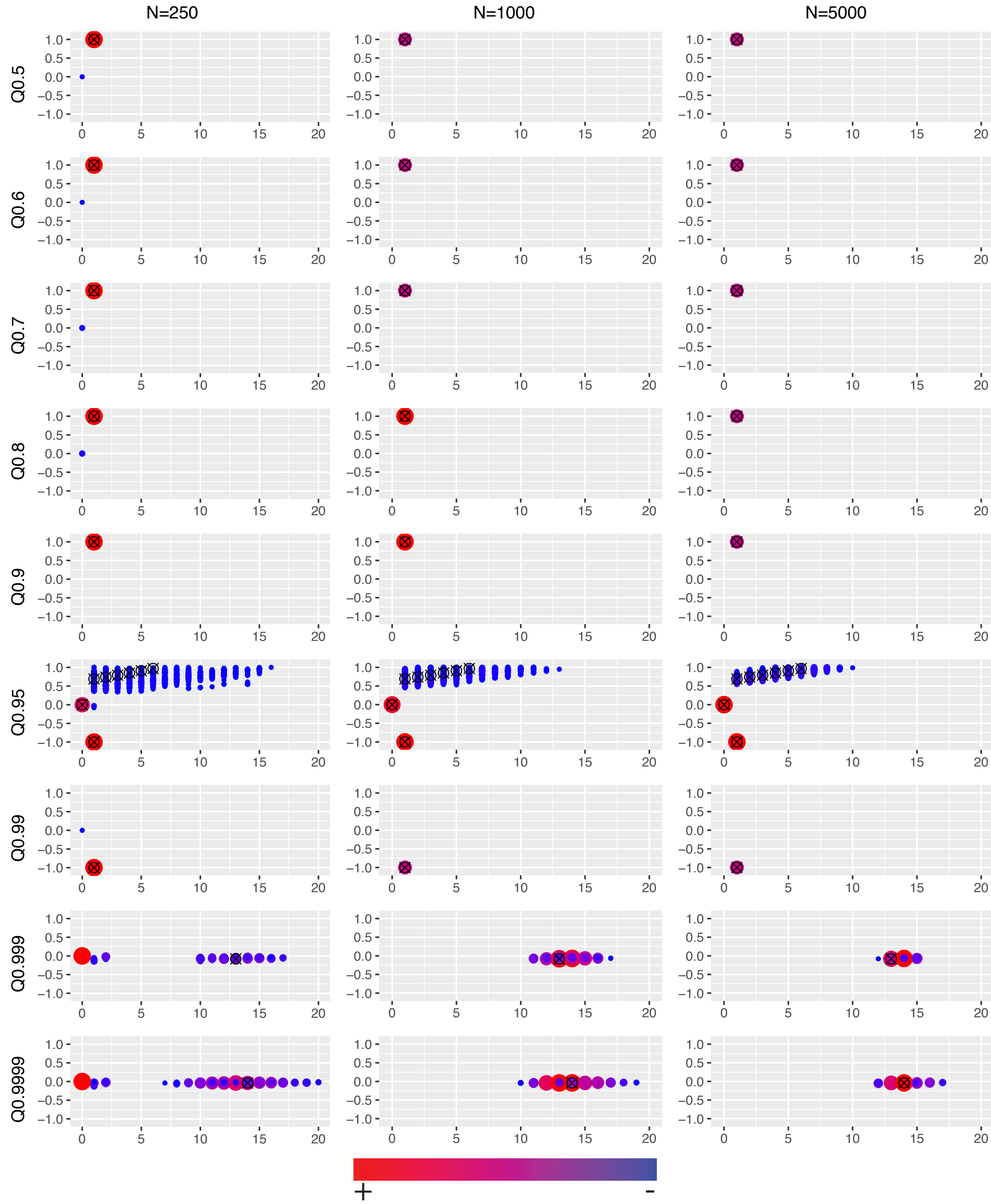
[5]The results are qualitatively similar to those obtained when using a non-causal $AR(1)$ as a DGP, but the latter seems to be quite restrictive in practice as it imposes a sudden crash of the bubble. We prefer the more general $MAR(1,1)$ specification and accept a loss in efficiency in the case where the causal parameter should actually be null.

Figure 2: Optimal portfolio strategies (lower quantiles)



Notes: Mass repartition of the optimal portfolio strategies for the CRRA utility function with $\gamma = 10$ when the speculative asset's parameters are estimated by ML across 2000 simulated trajectories of length $N = 250, 1000, 5000$ trading days and for several starting values defined by the quantiles, $Q_{\cdot}$, of the true marginal distribution of X_t. The DGP for the speculative asset price is a MAR(1,1) process $(1-0.9F)(1-0.1B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.3, 1.5, 15)$. The results are displayed in the share (vertical axis) - horizon (horizontal axis) space. The larger and redder the dots, the bigger the proportion of selected portfolios falling in that area across the 2000 simulations. A black target symbol indicates a *true* optimal portfolio, i.e. obtained for the true values of the parameters.

Figure 3: Optimal portfolio strategies (upper quantiles)



Notes: Mass repartition of the optimal portfolio strategies for the CRRA utility function with $\gamma = 10$ when the speculative asset's parameters are estimated by ML across 2000 simulated trajectories of length $N = 250, 1000, 5000$ trading days and for several starting values defined by the quantiles, $Q_{\cdot}$, of the true marginal distribution of X_t. The DGP for the speculative asset price is a MAR(1,1) process $(1-0.9F)(1-0.1B)X_t = \varepsilon_t$ with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.3, 1.5, 15)$. The results are displayed in the share (vertical axis) - horizon (horizontal axis) space. The larger and redder the dots, the bigger the proportion of selected portfolios falling in that area across the 2000 simulations. A black target symbol indicates a *true* optimal portfolio, i.e. obtained for the true values of the parameters.

3 Economic Value

This section includes additional simulation results on the economic value of the proposed portfolio allocation strategy, which are consistent with those in the mail paper. The BP strategies in Tables 1 to 6 in Subsection 3.1 are based on the first four moments of the return distribution of the bubble asset, while the BP2 strategies in Tables 7 to 12 (in Subsection 3.2) rely only on the first two moments. The average of the median terminal wealth for the three MAR(1,1)-based portfolios is in most cases well above that of the benchmark portfolios and superior to the invested amount. The only exception is the worse BP strategy, which in the PGP case is slightly inferior to 1. The positive averages of the opportunity cost also support these findings. Moreover, the performance of the BP2 strategies is dominated by that of BP, which is consistent with the DGP. Note also that in the BP2 case there is only one PGP strategy, as the fourth moment, which differentiated them before, is now irrelevant.

3.1 BP strategies based on the first four moments (BP)

Table 1: CRRA ($\gamma = 5$): Relative performance of portfolio strategies

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW	
Wealth	μ	$1.018^{*/*}$	1.005	$1.032^{*/*}$	1.003	1.002	
	σ	0.048	0.048	0.046	0.052	0.006	
			EW vs			MV vs	
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.011	-0.003	0.043	0.010	0.001	0.014
	σ	0.062	0.062	0.063	0.046	0.046	0.047

Notes: Our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth. The opportunity cost (OC) relatively to the two benchmark portfolios is also provided. The results take the form of out-of-sample average and standard deviation over the 1000 simulations. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilcoxon's test of equality of medians at the 95% level relatively to each of the two benchmark strategies, EW and MV, respectively.

Table 2: CRRA ($\gamma = 5$): Relative performance of portfolio strategies in positive bubble period

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV
Wealth	μ	$1.067^{*/*}$	$1.065^{*/*}$	$1.069^{*/*}$	1.031	1.019
	σ	0.021	0.021	0.021	0.030	0.020

		vs EW			vs MV		
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.025	0.025	0.025	0.012	0. 012	0.019
	σ	0.055	0.020	0.020	0.050	0.020	0.020

Notes: see note to Table 1. The results are based only on the cases where the investment is performed while the first asset exhibits a bubble period, i.e. $X_{T+k} = x$ is beyond the 95% quantile of the theoretical distribution of the price process.

Table 3: PGP(1,1,1,1): Relative performance of portfolio strategies

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW
Wealth	μ	$1.021^{*/*}$	0.995	$1.095^{*/*}$	1.003	1.002
	σ	0.093	0.017	0.019	0.052	0.006

Notes: see note to Table 1.

Table 4: PGP(1,1,1,1): Relative performance of portfolio strategies in positive bubble period

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV
Wealth	μ	$1.036^{*/*}$	$0.999^{*/*}$	$1.055^{*/*}$	1.031	1.019
	σ	0.038	0.008	0.011	0.030	0.020

Notes: see note to Table 2.

Table 5: PGP(1,1,1,4): Relative performance of portfolio strategies

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW
Wealth	μ	$1.022^{*/*}$	0.996	$1.101^{*/*}$	1.003	1.002
	σ	0.007	0.019	0.020	0.052	0.006

Notes: see note to Table 1.

Table 6: PGP(1,1,1,4): Relative performance of portfolio strategies in positive bubble period

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV
Wealth	μ	$1.044^{*/*}$	$0.999^{*/*}$	$1.071^{*/*}$	1.031	1.019
	σ	0.021	0.011	0.012	0.030	0.020

Notes: see note to Table 2.

3.2 BP strategies based on the first two moments (BP2)

Table 7: CRRA ($\gamma = 10$): Relative performance of BP2 portfolio strategies

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW	
Wealth	μ	$1.018^{*/*}$	1.006	$1.034^{*/*}$	1.005	1.001	
	σ	0.108	0.179	0.180	0.109	0.006	
			EW vs			MV vs	
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.013	0.005	0.027	0.016	0.011	0.028
	σ	0.112	0.131	0.133	0.096	0.135	0.177

Notes: Our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth. The opportunity cost (OC) relatively to the two benchmark portfolios is also provided. The results take the form of out-of-sample average and standard deviation over the 1000 simulations. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilcoxon's test of equality of medians at the 95% level relatively to each of the two benchmark strategies, EW and MV, respectively.

Table 8: CRRA ($\gamma = 10$): Relative performance of BP2 portfolio strategies in positive bubble period

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV	
Wealth	μ	$1.020^{*/*}$	$1.002^{*/*}$	$1.023^{*/*}$	1.005	1.001	
	σ	0.025	0.025	0.026	0.108	0.006	
			vs EW			vs MV	
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.034	0.031	0.036	0.039	0.032	0.046
	σ	0.099	0.099	0.010	0.025	0.026	0.025

Notes: see note to Table 7. The results are based only on the cases where the investment is performed while the first asset exhibits a bubble period, i.e. $X_{T+k} = x$ is beyond the 95% quantile of the theoretical distribution of the price process.

Table 9: CRRA ($\gamma = 5$): Relative performance of BP2 portfolio strategies

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW	
Wealth	μ	$1.018^{*/*}$	1.006	$1.034^{*/*}$	1.005	1.001	
	σ	0.108	0.179	0.180	0.109	0.006	
			EW vs			MV vs	
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.022	0.018	0.027	0.024	0.016	0.031
	σ	0.149	0.149	0.150	0.168	0.169	0.168

Notes: see note to Table 7.

Table 10: CRRA ($\gamma = 5$): Relative performance of BP2 portfolio strategies in positive bubble period

		Strategy					
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV	
Wealth	μ	$1.028^{*/*}$	$1.028^{*/*}$	$1.028^{*/*}$	1.005	1.001	
	σ	0.026	0.026	0.026	0.108	0.006	
			vs EW			vs MV	
		BP_{mean}	BP_{inf}	BP_{sup}	BP_{mean}	BP_{inf}	BP_{sup}
OC	μ	0.031	0.031	0.030	0.092	0.003	0.092
	σ	0.100	0.100	0.010	0.026	0.026	0.026

Notes: see note to Table 8.

Table 11: PGP : Relative performance of BP2 portfolio strategies

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	EW
Wealth	μ	$1.017^{*/*}$	0.986	$1.048^{*/*}$	1.005	1.001
	σ	0.067	0.078	0.078	0.109	0.006

Notes: see note to Table 7.

Table 12: PGP : Relative performance of BP2 portfolio strategies in positive bubble period

		Strategy				
		BP_{mean}	BP_{inf}	BP_{sup}	EW	MV
Wealth	μ	$1.019^{*/*}$	$1.000^{*/*}$	$1.033^{*/*}$	1.005	1.001
	σ	0.018	0.011	0.030	0.108	0.006

Notes: see note to Table 8.

4 Empirical Application

This section includes supplementary results on portfolio allocation with a bubble asset, where the latter is either the CO2, the Brent or the WTI price. Subsection 4.1 discusses the case of the Polynomial Goal Programming approach. At the same time, in Subsection 4.2 we analyze the case where one accounts for transaction costs, much relevant for any investor, and in Subsection 4.3 the case where the BP optimisation exploits only the information contained in the first two conditional moments, which puts our approach on a more equal footing with the benchmarks.

4.1 Polynomial Goal Programming (PGP) optimization algorithm

As in the CRRA case presented in the manuscript, we identify the optimal portfolio strategies under PGP based on the in-sample data. Table 13 reports these strateg(y/ies) for a range of quantiles in the upper half of the distribution of each of the three bubble assets. Notice that when the conditioning price is in the quantiles close to the center of the distribution, the investor takes almost no position as the weights are either 0 or very close to it. In contrast, when the investment takes place further in the bubble period, two types of comparable optimal – according to the PGP criterion defined – strategies arise: a long and a short one. The shorter one is characterized by a larger investment share in the bubble asset over a possibly slightly shorter horizon (down by one to five weeks depending on the portfolio and the sensitivity of the investor to kurtosis). The investor who weights more the fourth moment appears to choose more extreme allocations in the bubble asset for both long and short strategies. Another main difference with respect to the CRRA case is that the PGP investor does not take the risk to wait until the terminal horizon and see whether the bubble continues and hence does not short the positions beyond 19 trading weeks for the 0.99 quantile and above. Still, the more bubblier oil series are characterized by larger investment horizons than the CO2, as in the CRRA case.

Table 13: Optimal portfolio strategies under PGP

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
Panel A: CO2									
PGP (1,1,1,1) (w^*, h^*)	(-0.01,1) (0,0)	(-0.01,1) (0,0)	(-0.01,2) (0,0)	(-0.01,2) (0,0)	(-0.01,2) (0,0)	(-0.02,1) (0,0)	(-0.66,1) (0.32,2) (0,0)	(-0.54,1) (0.29,2) (0,0)	(-0.5,1) (0.29,2) (0,0)
PGP(1,1,1,4) (w^*, h^*)	(0,0)	(0,0)	(0,0)	(-0.01,1) (0,0)	(-0.01,2) (0,0)	(-0.01,2) (0,0)	(-0.95,1) (0.57,1) (0,0)	(-0.79,1) (0.56,1)	(-0.73,1) (0.52,1)
Panel B: Brent									
PGP (1,1,1,1) (w^*, h^*)	(-0.01,2) (0.01,2) (0,0)	(-0.02,2) (-0.01,4) (0,0)	(-0.02,3) (-0.01,6) (0,0)	(-0.02,4) (-0.01,7) (0,0)	(-0.03,3) (-0.01,7) (0,0)	(-0.04,3) (0,0)	(-0.76,3) (0.38,6)	(-0.58,3) (0.31,6)	(-0.53,3) (0.28,6)
PGP(1,1,1,4) (w^*, h^*)	(0,0)	(0,0)	(-0.01,6) (0,0)	(-0.01,7) (0,0)	(-0.01,7) (0,0)	(-0.01,7) (0,0)	(-0.95,4) (0.63,5)	(-0.73,4) (0.51,5)	(-0.66,4) (0.46,5)
Panel C: WTI									
PGP (1,1,1,1)	(-0.02,3) (0.01,3) (0,0)	(-0.03,4) (0,0)	(-0.04,4) (0,0)	(-0.05,5) (0,0)	(-0.05,6) (0,0)	(-0.06,6) (-0.01,20) (0,0)	(-0.11,6) (-0.01,17) (0,0)	(-0.71,6) (0.38,11)	(-0.67,6) (0.36,11)
PGP(1,1,1,4)	(0.01,21) (-0.01,18) (-0.02,12) (0,0)	(-0.02,14) (-0.01,20) (0.01,22) (0.02,14) (0,0)	(-0.02,15) (-0.03,12) (-0.01,22) (0.01,21) (0.02,14) (0,0)	(-0.03,13) (-0.02,16) (-0.01,23) (0,0)	(-0.03,13) (-0.02,16) (-0.01,24) (0,0)	(-0.03,12) (-0.02,16) (-0.01,23) (0,0)	(-0.06,9) (-0.01,19) (0,0)	(0.63,8) (-0.94,8)	(0.6,8) (-0.88,8)

Notes: The table displays the optimal portfolio strategies for each portfolio conditional on the quantile of the in-sample distribution in which the bubble asset is at the time of the investment. w is reported in percentages of the investment and h in weeks.

To gauge the performance of our MAR(1,1)-based approach using the optimal strategies under PGP, we proceed similarly to the case of the CRRA utility function. Table 14 reports the summary statistics including the average over the out-of-sample of the median and standard deviation of the number of rebalancings and of the number of investment paths. Similarly to the CRRA case, the CO2-EUREX Euro bund portfolio is still the most often rebalanced, while the WTI-US T-bond is the least rebalanced one with small variations from one quantile to another. Note also that the most dense tree of investment paths under

PGP corresponds to the CO2-EUREX Euro bund portfolio irrespective of the sensitivity of the investor to the kurtosis. This is most certainly due to the fact that the optimal investment horizon h is at most equal to 2 while in the upper quantiles the investor can chose each time between a long and a short strategy.

Table 14: Number of rebalancings and strategies

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
Panel A: CO2						
PGP (1,1,1,1)	$Med_{N^\circ Rebalancings}$	7.000	6.662	7.125	26.000	26.000
	$\sigma_{N^\circ Rebalancings}$	3.928	3.788	4.191	0	0
	$Med_{N^\circ Paths}$	8	76	8	1	1
	$\sigma_{N^\circ Paths}$	275.071	2750.47	275.071	0	0
PGP (1,1,1,4)	$Med_{N^\circ Rebalancings}$	6.000	6.375	6.2	26.000	26.000
	$\sigma_{N^\circ Rebalancings}$	3.382	3.341	3.343	0	0
	$Med_{N^\circ Paths}$	4	40	4	1	1
	$\sigma_{N^\circ Paths}$	120.032	1200.957	120.032	0	0
Panel B: Brent						
PGP (1,1,1,1)	$Med_{N^\circ Rebalancings}$	7.000	6.549	7.063	26.000	26.000
	$\sigma_{N^\circ Rebalancings}$	1.341	1.156	1.382	0	0
	$Med_{N^\circ Paths}$	9	85	9	1	1
	$\sigma_{N^\circ Paths}$	2.689	27.673	2.689	0	0
PGP (1,1,1,4)	$Med_{N^\circ Rebalancings}$	6.000	6.098	6.200	26.000	26.000
	$\sigma_{N^\circ Rebalancings}$	2.050	1.981	2.139	0	0
	$Med_{N^\circ Paths}$	3	29.5	3	1	1
	$\sigma_{N^\circ Paths}$	1.517	17.33	1.517	0	0
Panel C: WTI						
PGP (1,1,1,1)	$Med_{N^\circ Rebalancings}$	3.000	2.75	3.000	26.000	26.000
	$\sigma_{N^\circ Rebalancings}$	0.978	0.337	1.138	0	0
	$Med_{N^\circ Paths}$	1	7	1	1	1
	$\sigma_{N^\circ Paths}$	0.308	2.758	0.308	0	0
PGP (1,1,1,4)	$Med_{N^\circ Rebalancings}$	2.000	1.667	1.000	26.000	26.000
	$\sigma_{N^\circ Rebalancings}$	0.43	0.24	0.732	0	0
	$Med_{N^\circ Paths}$	1	3	1	1	1
	$\sigma_{N^\circ Paths}$	0	1.057	0	0	0

Notes: The table displays the average over the out-of-sample of the median and the standard deviation of the number of rebalancings and optimal portfolio trajectories in the 10% best ($BP_{90\%}$) and worst ($BP_{10\%}$) performing BP strategies as well as over the full sample of BP strategies ($BP_{50\%}$) with horizon H.

We evaluate the out-of-sample performance of the BP strategies under PGP through

Table 15: Relative performance of portfolio strategies under PGP: terminal wealth

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.992^{/***}$	$1.061^{*/***}$	$1.190^{***/***}$	1.049	0.922
	σ	0.075	0.078	0.178	0.140	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$0.976^{**/***}$	$1.049^{/***}$	$1.149^{***/***}$	1.049	0.922
	σ	0.086	0.076	0.133	0.140	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.978^{/***}$	$1.055^{*/***}$	$1.203^{***/***}$	1.001	0.921
	σ	0.090	0.080	0.159	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$0.961^{*/**}$	$1.050^{*/***}$	$1.193^{***/***}$	1.001	0.921
	σ	0.099	0.087	0.151	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.983^{/**}$	$1.061^{**/***}$	$1.137^{***/***}$	0.990	0.932
	σ	0.168	0.094	0.155	0.167	0.056
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$1.065^{***/***}$	$1.068^{***/***}$	$1.080^{***/***}$	0.990	0.932
	σ	0.081	0.081	0.098	0.167	0.056

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth. The results take the form of out-of-sample average and standard deviation. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilkoxon's test at the 90%, 95% and 99% levels.

the terminal wealth criterion. The results are reported in Table 15.[6] The distribution of the wealth seems to be more spread than in the CRRA case, with a $BP_{90\%}$ strategy similar to that in Table 8 of the main paper but with a much inferior $BP_{10\%}$ strategy.

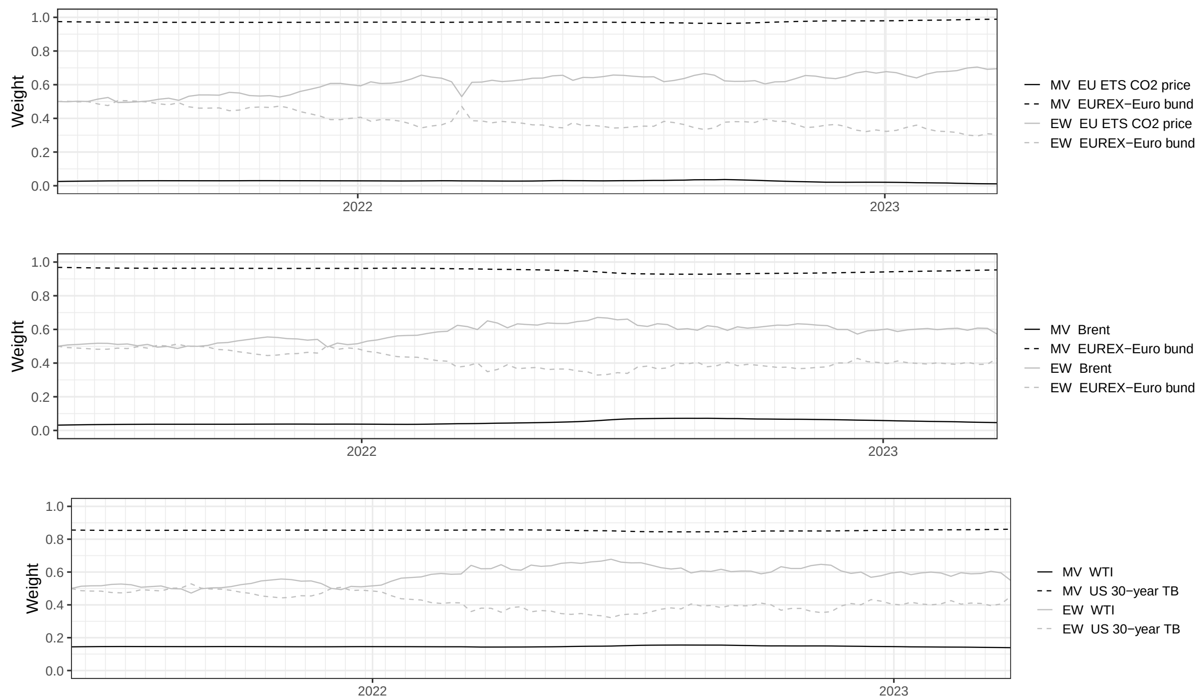
Still, the results indicate a positive gain in using our approach relatively to the standard MV and EW portfolios. Indeed, the return on investment is positive in all three panels for the median and upper quantiles of the BP approach and both PGP specifications. In contrast, the EW strategy beats $BP_{10\%}$.

[6] As the PGP approach is not utility-based, we cannot rely on the opportunity cost as a performance measure anymore.

4.2 Transaction costs

We gauge the impact of transaction costs on the performance of our portfolio strategies relative to the benchmarks in a simple setup where transaction costs are fixed at 0.05% per unit of investment. The results for the CRRA approach are displayed in Tables 16 and 17, while those for PGP are reported in Table 18. They support the main findings of the paper, in particular the superiority of the BP approach over the two benchmarks.

Figure 4: EW and MV weights



In presence of transaction costs, the BP strategies generally register a slightly larger drop in terminal wealth than the benchmarks: 0.010 vs 0.003 on average. This is specific to our setup of transaction costs that are proportional to the investment in each asset. Indeed, although our approach involves sparse rebalancing, the quantities exchanged in each trade are larger than the sum of those rebalanced daily by the benchmark portfolios. This can be easily seen if one compares w in Tables 6 of the paper and 13 above with the dynamics of the weights of the benchmarks displayed in Figure 4. The latter depicts the weights of the benchmark models obtained within a one-period-ahead rolling window scheme. Their slow-varying behaviour suggests very low ajustements of the portfolio and thus very low proportional transaction costs.

However, even in presence of transaction costs the results indicate a positive gain in using the BP approach relative to the two benchmarks. It is only in the case of the 10% worst strategies that the opportunity cost is negative and the *EW* portfolio is preferred.

Table 16: Relative performance of portfolio strategies under CRRA with transaction costs

Panel A: CO_2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	1.048$^{/***}$	1.059$^{/***}$	1.070$^{*/***}$	1.045	0.922
	σ	0.103	0.115	0.125	0.139	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	0.984$^{/***}$	1.130$^{***/***}$	1.345$^{***/***}$	1.045	0.922
	σ	0.071	0.103	0.135	0.139	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	1.032$^{/***}$	1.038$^{/***}$	1.061$^{**/***}$	0.998	0.921
	σ	0.073	0.078	0.111	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	0.983$^{/***}$	1.074$^{***/***}$	1.221$^{***/***}$	0.998	0.921
	σ	0.080	0.099	0.188	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	1.190$^{***/***}$	1.190$^{***/***}$	1.191$^{***/***}$	0.987	0.932
	σ	0.176	0.176	0.175	0.167	0.055
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	1.135$^{***/***}$	1.216$^{***/***}$	1.302$^{***/***}$	0.987	0.932
	σ	0.188	0.176	0.177	0.167	0.055

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth. The results take the form of out-of-sample average and standard deviation. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilkoxon's test at the 90%, 95% and 99% levels. The transaction costs are fixed at 0.05% per unit of investment.

Table 17: Relative performance of portfolio strategies under CRRA: opportunity cost (with transaction costs)

Panel A: CO2							
			EW vs			MV vs	
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.124	0.013	0.024	0.126	0.137	0.148
	σ	0.110	0.218	0.226	0.125	0.136	0.147
CRRA ($\gamma = 10$)	μ	-0.124	0.085	0.300	0.062	0.208	0.424
	σ	0.110	0.155	0.179	0.083	0.113	0.142

Panel B: Brent							
			EW vs			MV vs	
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.077	0.041	0.063	0.111	0.118	0.140
	σ	0.142	0.220	0.240	0.100	0.104	0.132
CRRA ($\gamma = 10$)	μ	-0.077	0.076	0.223	0.062	0.153	0.300
	σ	0.142	0.201	0.292	0.105	0.126	0.212

Panel C: WTI							
			EW vs			MV vs	
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.055	0.203	0.203	0.258	0.258	0.259
	σ	0.132	0.313	0.313	0.219	0.219	0.218
CRRA ($\gamma = 10$)	μ	-0.055	0.229	0.315	0.203	0.284	0.370
	σ	0.132	0.248	0.246	0.205	0.186	0.191

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of opportunity cost (OC). Transaction costs are fixed at 0.05% per unit of investment.

Table 18: Relative performance of portfolio strategies under PGP with transaction costs

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.984^{/***}$	$1.051^{*/***}$	$1.177^{***/***}$	1.045	0.922
	σ	0.075	0.077	0.174	0.139	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$0.968^{**/***}$	$1.041^{/***}$	$1.139^{***/***}$	1.045	0.922
	σ	0.086	0.075	0.130	0.139	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.968^{/***}$	$1.045^{*/***}$	$1.19^{***/***}$	0.998	0.921
	σ	0.090	0.079	0.155	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$0.952^{*/**}$	$1.041^{*/***}$	$1.181^{***/***}$	0.998	0.921
	σ	0.100	0.087	0.147	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	μ	$0.974^{/**}$	$1.052^{**/***}$	$1.127^{***/***}$	0.987	0.932
	σ	0.168	0.094	0.152	0.167	0.055
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	μ	$1.056^{***/***}$	$1.059^{***/***}$	$1.072^{***/***}$	0.987	0.932
	σ	0.081	0.080	0.096	0.167	0.055

Notes: see note to Table 16. The transaction costs are fixed at 0.05% per unit of investment.

4.3 First two conditional moments

This setup allows one to study the performance of the bubble portfolio strategies (labeled BP2) when only the first two conditional moments are taken into account for each of the three assets. The bubble-asset dynamics is still assumed to follow a MAR(1,1) process whose estimated parameters are reported in Table 4 of the paper. However, the CRRA utility maximization program in equation (2) and the PGP model in equation (B.1) rely only on the first two conditional moments of the distribution of the terminal wealth in this setup. By comparison with the results of the main paper, the findings here are expected to shed light on the contribution of third and fourth conditional moments to the performance of the bubble portfolio strategies.

The resulting optimal portfolio strateg(y/ies) in the form of couples (ω, h) are reported in Tables 19 and 25, respectively, while the number of rebalancings are displayed in Tables 20 and 26. They highlight shorter positions on longer horizons and less rebalancing than in the BP case.

Finally, the performance results are reported in Tables 21 - 24, 27 and 28. Note that in the BP2 case there is only one PGP strategy, as the fourth moment, which differentiated them before, is now irrelevant. The performance of BP2 strategies seems to be slightly better than that of BP for the CO2 data, but the results are more mitigated for the other two bubble series. We therefore suggest one to rely on the more general four-moments based approach and use BP2 for sensitivity analysis.

Table 19: Optimal BP2 portfolio strategies under CRRA

Panel A: CO2

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 5$) (w^*, h^*)	(0.01,1) (0,0)	(-0.04,7) (0,0)	(-0.1,7) (0,0)	(-0.19,7)	(-1,7) (-0.41,7)	(-1,8)	(-0.16,7) (-1,8)	(-0.1,7) (-1,8)	(-1,8) (-0.09,7)

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 10$) (w^*, h^*)	(0.01,1) (0,0)	(-0.02,8) (0,0)	(-0.05,7) (0,0)	(-0.09,7)	(-1,9)	(-1,10)	(-1,10)	(-1,11)	(-1,11)

Panel B: Brent

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 5$) (w^*, h^*)	(0.07,1) (0.02,18) (0,0)	(-0.03,26) (0.03,1) (0,0)	(-0.1,24) (0,0)	(-0.19,24)	(-0.33,24)	(-1,25) (-0.42,24)	(-1,26) (-0.19,24)	(-1,26) (-0.11,24)	(-1,26)

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 10$) (w^*, h^*)	(0.06,1) (0.01,18) (0,0)	(-0.01,26) (-0.02,24) (0.05,1) (0,0)	(-0.05,25) (0.02,1) (0,0)	(-0.09,25)	(-1,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)

Panel C: WTI

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 5$) (w^*, h^*)	(0.02,1) (0.01,26) (0,0)	(-0.02,26) (0.02,1) (0,0)	(-0.08,26) (0.02,1) (0,0)	(-0.22,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
CRRA ($\gamma = 10$) (w^*, h^*)	(0.02,1) (0,0)	(-0.01,26) (0.03,1) (0,0)	(-0.04,26) (0.04,1)	(-1,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)	(-1,26)

Notes: The Table displays the optimal portfolio strategies (w^*, h^*) for each portfolio conditional on the quantile of the in-sample distribution in which the bubble asset is at the time of the investment. w is reported in percentages of the investment and h in weeks.

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Table 20: Number of rebalancings and BP2 strategies under CRRA

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
Panel A: CO2						
CRRA ($\gamma = 5$)	$\mu_{N^o Rebalancings}$	3.000	3.000	3.000	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.611	0.471	0.642	0	0
	$\mu_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.653	0	0	0
CRRA ($\gamma = 10$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$\mu_{N^o Rebalancings}$	3.462	3.333	3.333	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.813	0.572	0.461	0	0
	$\mu_{N^o Paths}$	10	101	10	1	1
	$\sigma_{N^o Paths}$	17.206	181.978	17.215	0	0
Panel B: Brent						
CRRA ($\gamma = 5$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$\mu_{N^o Rebalancings}$	2.000	2 .000	2.000	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.173	0.085	0.121	0	0
	$\mu_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.398	0	0	0
CRRA ($\gamma = 10$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$\mu_{N^o Rebalancings}$	2.000	2 .000	2.000	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.405	0.364	0.642	0	0
	$\mu_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	2.817	29.92	2.817	0	0
Panel C: WTI						
CRRA ($\gamma = 5$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$\mu_{N^o Rebalancings}$	2.000	2.000	2.000	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.171	0.149	0.606	0	0
	$\mu_{N^o Paths}$	1	1	1	1	1
	$\sigma_{N^o Paths}$	0	0.813	0	0	0
CRRA ($\gamma = 10$)		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
	$\mu_{N^o Rebalancings}$	2.000	3.333	3.333	26.000	26.000
	$\sigma_{N^o Rebalancings}$	0.593	0.572	0.461	0	0
	$\mu_{N^o Paths}$	1	101	10	1	1
	$\sigma_{N^o Paths}$	0.708	181.978	17.215	0	0

Notes: The table displays the average over the out-of-sample of the median and the standard deviation of the number of rebalancings and portfolio trajectories in the 10% best (resp. worst) performing BP strategies, $BP_{90\%}$ (resp. $BP_{10\%}$) as well as over the full sample of MAR strategies with horizon H ($BP_{50\%}$).

Table 21: Relative performance of BP2 portfolio strategies under CRRA (without transaction costs)

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$0.999^{*/***}$	$1.007^{/***}$	$1.026^{/***}$	1.049	0.922
	σ	0.234	0.236	0.247	0.140	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$0.939^{***/***}$	$1.116^{***/***}$	$1.346^{***/***}$	1.049	0.922
	σ	0.122	0.119	0.132	0.140	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.192^{***/***}$	$1.192^{***/***}$	$1.194^{***/***}$	1.001	0.921
	σ	0.175	0.174	0.173	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.108^{***/***}$	$1.154^{***/***}$	$1.230^{***/***}$	1.001	0.921
	σ	0.166	0.157	0.189	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.247^{***/***}$	$1.248^{***/***}$	$1.253^{***/***}$	0.990	0.932
	σ	0.194	0.193	0.189	0.167	0.056
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.146^{***/***}$	$1.246^{***/***}$	$1.325^{***/***}$	0.990	0.932
	σ	0.188	0.186	0.190	0.167	0.056

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth. The results take the form of out-of-sample average and standard deviation. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilkoxon's test at the 90%, 95% and 99% levels.

Table 22: Relative performance of BP2 portfolio strategies under CRRA (with transaction costs)

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$0.988^{*/***}$	$0.996^{/***}$	$1.015^{/***}$	1.045	0.922
	σ	0.231	0.234	0.244	0.139	0.046
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$0.928^{***/***}$	$1.099^{***/***}$	$1.326^{***/***}$	1.045	0.922
	σ	0.122	0.113	0.131	0.139	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.181^{***/***}$	$1.182^{***/***}$	$1.183^{***/***}$	0.998	0.921
	σ	0.174	0.174	0.172	0.152	0.044
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.098^{***/***}$	$1.143^{***/***}$	$1.218^{***/***}$	0.998	0.921
	σ	0.165	0.156	0.187	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 5$)	μ	$1.236^{***/***}$	$1.237^{***/***}$	$1.242^{***/***}$	0.987	0.932
	σ	0.193	0.192	0.188	0.167	0.055
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
CRRA ($\gamma = 10$)	μ	$1.136^{***/***}$	$1.235^{***/***}$	$1.314^{***/***}$	0.987	0.932
	σ	0.187	0.185	0.188	0.167	0.055

Notes: see note to Table 21. The transaction costs are fixed at 0.05% per unit of investment.

Table 23: Relative performance of BP2 portfolio strategies under CRRA: opportunity cost

Panel A: CO_2

		EW vs			MV vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.050	-0.042	-0.023	0.077	0.085	0.104
	σ	0.323	0.325	0.334	0.249	0.251	0.262
CRRA ($\gamma = 10$)	μ	-0.110	0.067	0.297	0.017	0.194	0.424
	σ	0.201	0.180	0.167	0.134	0.131	0.132

Panel B: Brent

		EW vs			MV vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.191	0.191	0.193	0.271	0.271	0.273
	σ	0.302	0.302	0.302	0.204	0.204	0.203
CRRA ($\gamma = 10$)	μ	0.107	0.153	0.229	0.187	0.233	0.309
	σ	0.258	0.256	0.294	0.191	0.185	0.217

Panel C: WTI

		EW vs			MV vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.257	0.257	0.263	0.315	0.316	0.321
	σ	0.336	0.335	0.336	0.238	0.237	0.236
CRRA ($\gamma = 10$)	μ	0.156	0.255	0.335	0.214	0.314	0.394
	σ	0.285	0.239	0.250	0.205	0.190	0.202

Notes: The quantiles of our MAR(1,1)-based strategies are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of opportunity cost (OC).

Table 24: Relative performance of BP2 portfolio strategies under CRRA: opportunity cost (with transaction costs)

Panel A: CO2

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.124	-0.049	-0.003	0.067	0.074	0.093
	σ	0.110	0.322	0.331	0.246	0.248	0.259
CRRA ($\gamma = 10$)	μ	-0.124	0.054	0.281	0.007	0.178	0.404
	σ	0.110	0.175	0.163	0.134	0.123	0.130

Panel B: Brent

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	-0.077	0.184	0.186	0.261	0.261	0.263
	σ	0.142	0.301	0.301	0.203	0.203	0.202
CRRA ($\gamma = 10$)	μ	-0.077	0.145	0.220	0.177	0.222	0.297
	σ	0.142	0.255	0.292	0.190	0.184	0.215

Panel C: WTI

		EW vs			*MV* vs		
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$
CRRA ($\gamma = 5$)	μ	0.191	0.191	0.193	0.271	0.271	0.273
	σ	0.302	0.302	0.302	0.204	0.204	0.203
CRRA ($\gamma = 10$)	μ	0.107	0.153	0.229	0.187	0.233	0.309
	σ	0.258	0.256	0.294	0.191	0.185	0.217

Notes: see note to Table 23. Transaction costs are fixed at 0.05% per unit of investment.

Table 25: Optimal BP2 portfolio strategies under PGP

Panel A: CO2

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP	(-0.01,7)	(-0.11,7)	(-0.23,7)	(-0.38,7)	(-0.55,7)	(-0.54,7)	(-0.28,7)	(-0.18,7)	(-0.17,7)
(w^*, h^*)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Panel B: Brent

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP	(0.04,23)	(-0.09,23)	(-0.23,24)	(-0.4,23)	(-0.55,23)	(-0.54,23)	(-0.31,24)	(-0.2,23)	(-0.18,24)
(w^*, h^*)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Panel C: WTI

$X_t = x$	$q_{0.5}$	$q_{0.6}$	$q_{0.7}$	$q_{0.8}$	$q_{0.9}$	$q_{0.95}$	$q_{0.99}$	$q_{0.999}$	$q_{0.9999}$
PGP	(0.01,26)	(-0.06,26)	(-0.18,26)	(-0.37,26)	(-0.69,26)	(-0.95,26)	(-0.97,26)	(-0.37,26)	(-0.35,26)
	(0,0)	(0,0)	(0,0)	(0,0)				(0,0)	(0,0)

Notes: The Table displays the optimal portfolio strategies (w^*, h^*) for each portfolio conditional on the quantile of the in-sample distribution in which the bubble asset is at the time of the investment. w is reported in percentages of the investment and h in weeks.

Table 26: Number of rebalancings and BP2 strategies under PGP

Panel A: CO2

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	$Med_{N^\circ Rebalancings}$	4	4	4	67	67
	$\sigma_{N^\circ Rebalancings}$	0	0	0	0	0
	$Med_{N^\circ Paths}$	1	1	1	1	1
	$\sigma_{N^\circ Paths}$	0	0	0	0	0
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	$Med_{N^\circ Rebalancings}$	4	4	4	67	67
	$\sigma_{N^\circ Rebalancings}$	0	0	0	0	0
	$Med_{N^\circ Paths}$	1	1	1	1	1
	$\sigma_{N^\circ Paths}$	0	0	0	0	0

Panel B: Brent

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	$Med_{N^\circ Rebalancings}$	2	1.5	1	68	68
	$\sigma_{N^\circ Rebalancings}$	0.477	0	0.477	0	0
	$Med_{N^\circ Paths}$	1	2	1	1	1
	$\sigma_{N^\circ Paths}$	0	0	0	0	0
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	$Med_{N^\circ Rebalancings}$	2	1.5	1	68	68
	$\sigma_{N^\circ Rebalancings}$	0.477	0	0.477	0	0
	$Med_{N^\circ Paths}$	1	2	1	1	1
	$\sigma_{N^\circ Paths}$	0	0	0	0	0

Panel C: WTI

		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,1)	$Med_{N^\circ Rebalancings}$	2	2	2	67	67
	$\sigma_{N^\circ Rebalancings}$	0.386	0.215	0.239	0	0
	$Med_{N^\circ Paths}$	1	1	1	1	1
	$\sigma_{N^\circ Paths}$	0	0.43	0	0	0
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP (1,1,1,4)	$Med_{N^\circ Rebalancings}$	2	2	2	67	67
	$\sigma_{N^\circ Rebalancings}$	0.386	0.215	0.239	0	0
	$Med_{N^\circ Paths}$	1	1	1	1	1
	$\sigma_{N^\circ Paths}$	0	0.43	0	0	0

Notes: The table displays the average over the out-of-sample of the median and the standard deviation of the number of rebalancings and portfolio trajectories in the 10% best (resp. worst) performing BP strategies, $BP_{90\%}$ (resp. $BP_{10\%}$) as well as over the full sample of MAR strategies with horizon H ($BP_{50\%}$).

Table 27: Relative performance of BP2 portfolio strategies under PGP (without transaction costs)

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.024^{/***}$	$1.024^{/***}$	$1.024^{/***}$	1.049	0.922
	σ	0.088	0.088	0.088	0.140	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.024^{/***}$	$1.038^{/***}$	$1.052^{*/***}$	1.001	0.921
	σ	0.090	0.081	0.074	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.229^{***/***}$	$1.240^{***/***}$	$1.252^{***/***}$	0.990	0.932
	σ	0.191	0.187	0.188	0.167	0.056

Notes: The quantiles of our MAR(1,1)-based strategies with first two moments are compared with the equally weighted (EW) and mean-variance (MV) ones in terms of terminal wealth. The results take the form of out-of-sample average and standard deviation. Asteriks ($*/*$) indicate the rejection of the null hypothesis of Wilkoxon's test at the 90%, 95% and 99% levels.

Table 28: Relative performance of BP2 portfolio strategies under PGP (with transaction costs)

Panel A: CO2						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.014^{/***}$	$1.014^{/***}$	$1.014^{/***}$	1.045	0.922
	σ	0.088	0.088	0.088	0.139	0.046
Panel B: Brent						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.016^{/***}$	$1.030^{/***}$	$1.045^{*/***}$	0.998	0.921
	σ	0.091	0.081	0.072	0.152	0.044
Panel C: WTI						
		$BP_{10\%}$	$BP_{50\%}$	$BP_{90\%}$	EW	MV
PGP	μ	$1.219^{***/***}$	$1.230^{***/***}$	$1.241^{***/***}$	0.987	0.932
	σ	0.189	0.186	0.187	0.167	0.055

Notes: see note to Table 27. The transaction costs are fixed at 0.05% per unit of investment.